Projection Based Model Order Reduction for Multiphysical Problems

Short Course Part 2 – I | Vector Spaces and Subspace Generation

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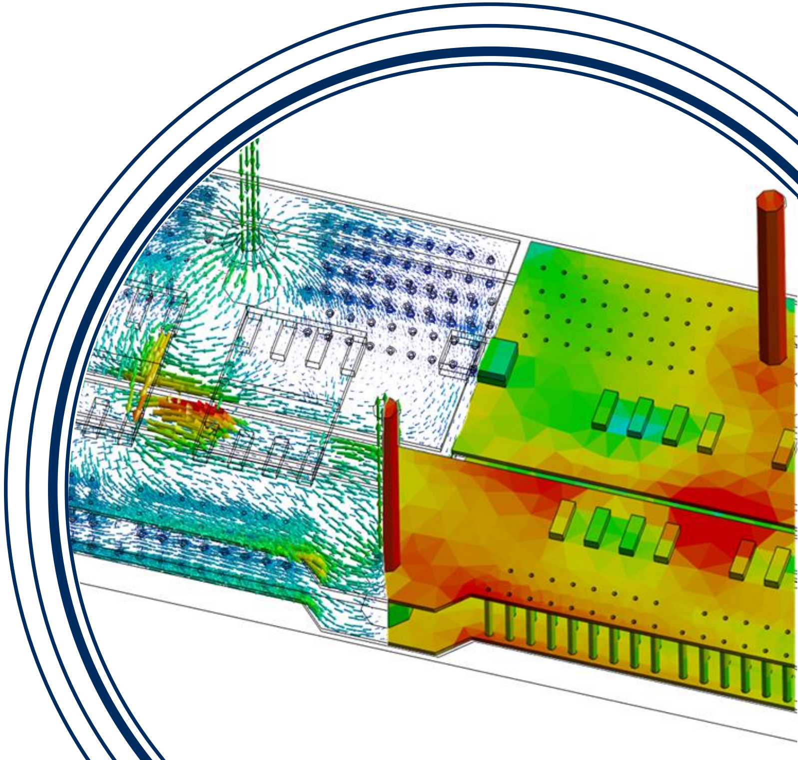
CADFEM Germany GmbH



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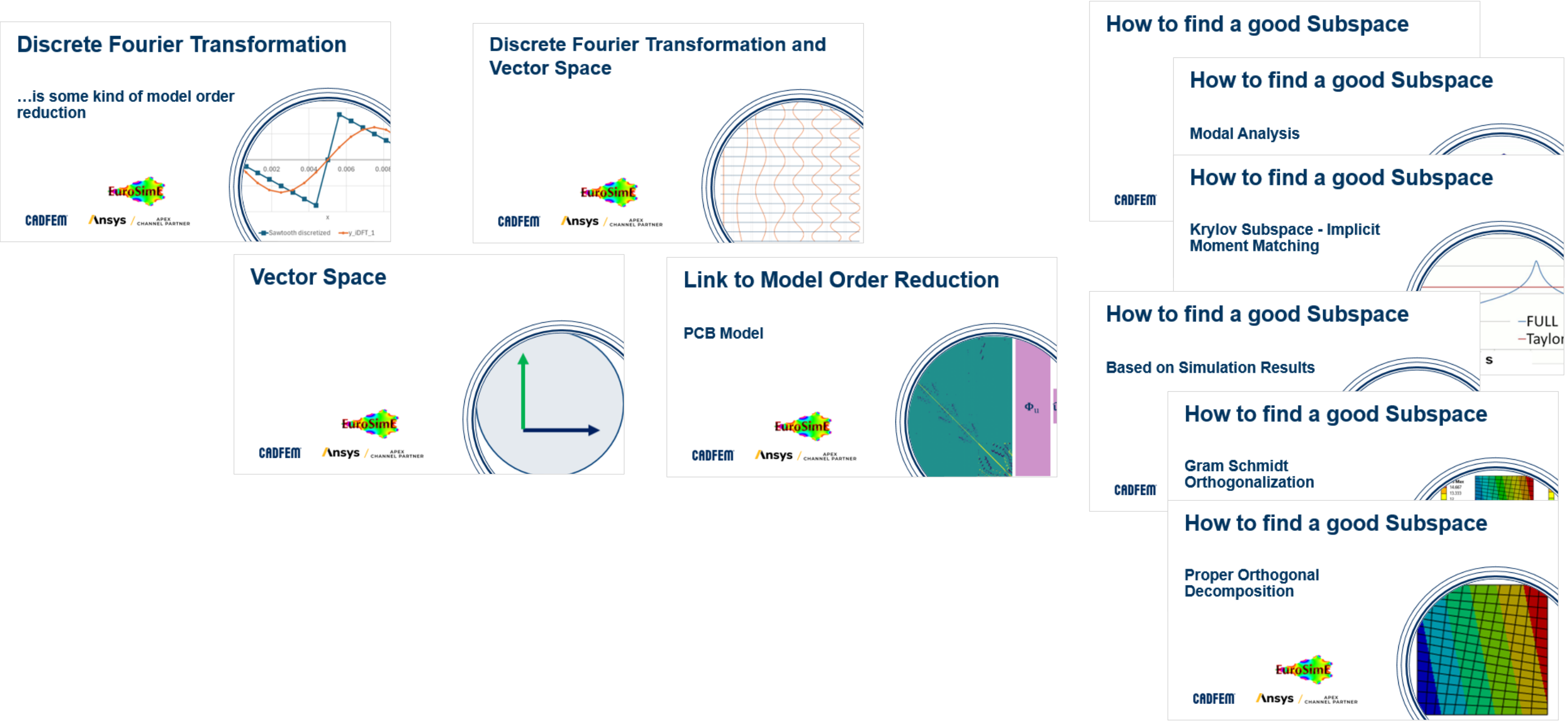
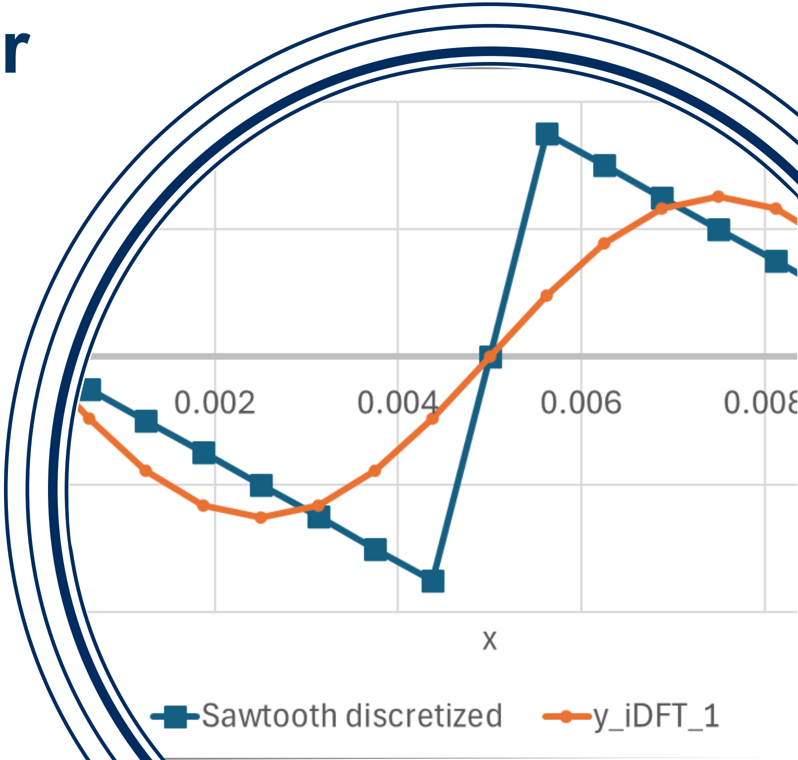
Agenda



Discrete Fourier Transformation
…is some kind of model order reduction
Discrete Fourier Transformation and Vector Space
Vector Space
Link to Model Order Reduction
PCB Model
How to find a good Subspace
Modal Analysis
Krylov Subspace - Implicit Moment Matching
Based on Simulation Results
Gram Schmidt Orthogonalization
Proper Orthogonal Decomposition

Discrete Fourier Transformation

…is some kind of model order reduction



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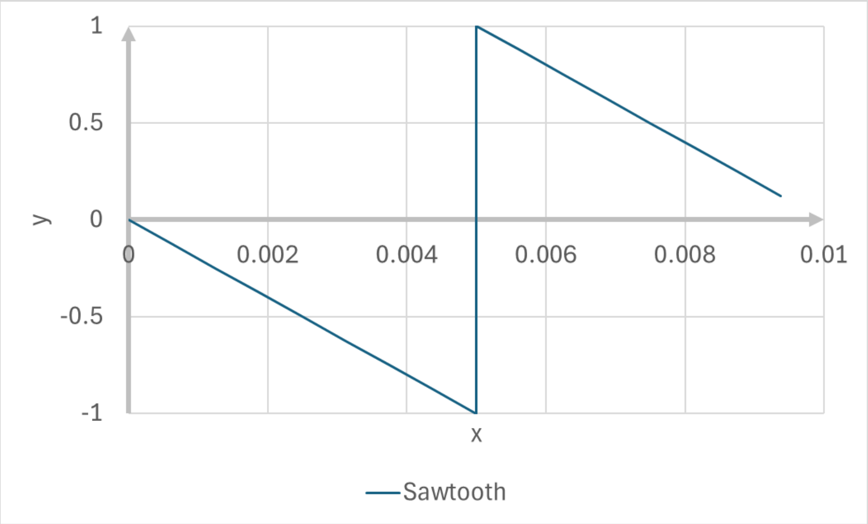
Discrete Fourier Transformation

Starting point

- Continuous series $y(x) \in \mathbb{R}^{\infty}$

→ Infinite number of points



Discrete Fourier Transformation

Discretization

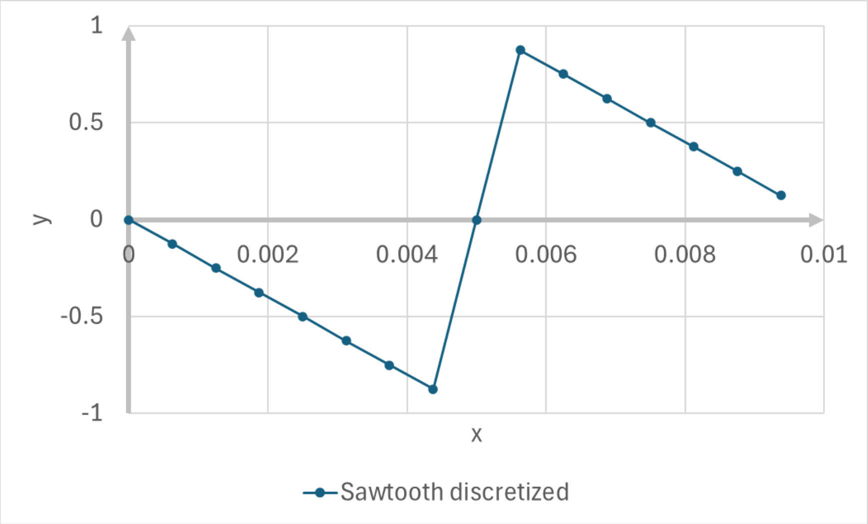
Starting point

- Continuous series $y(x) \in \mathbb{R}^{\infty}$
→ Infinite number of points

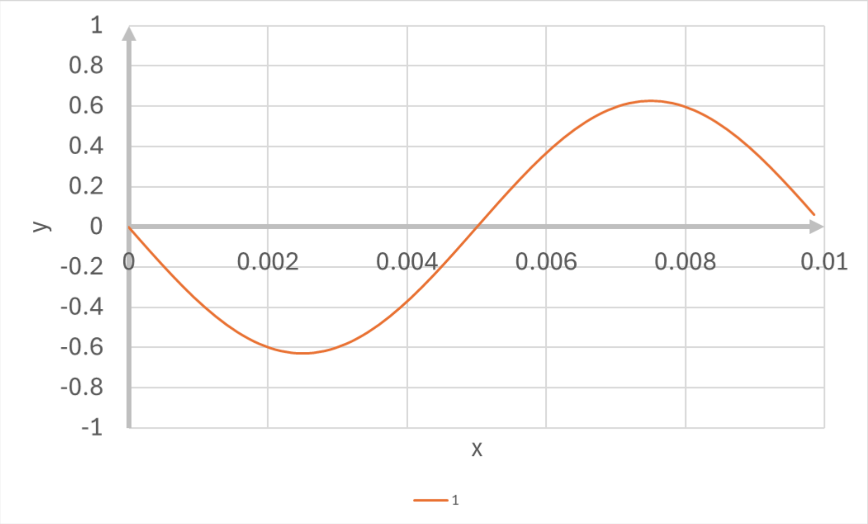
Discretized

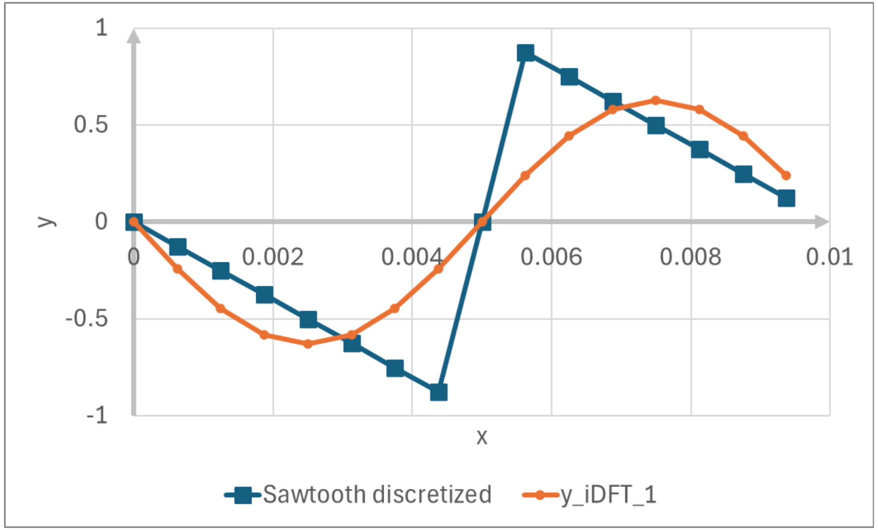
- $\mathbf{y} = \{y_0, \dots, y_{N-1}\}\ y \in \mathbb{R}^N$
- N Number of timepoints

→ From infinite to N ☺

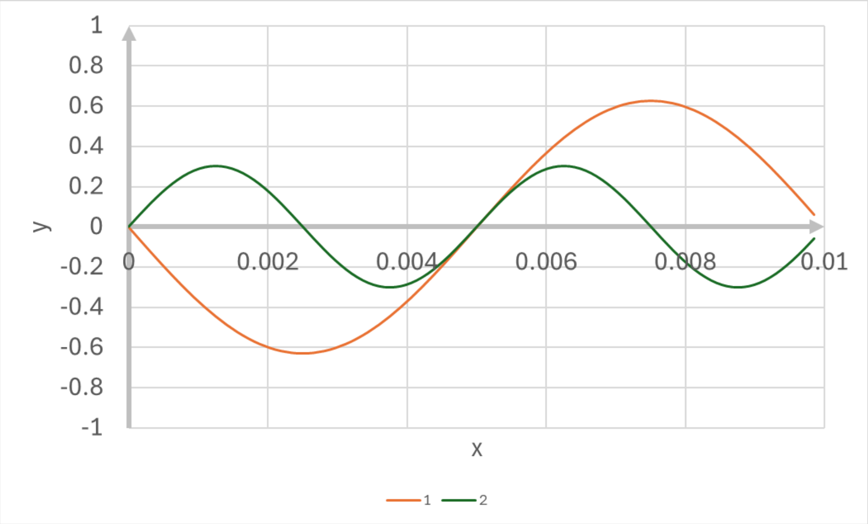


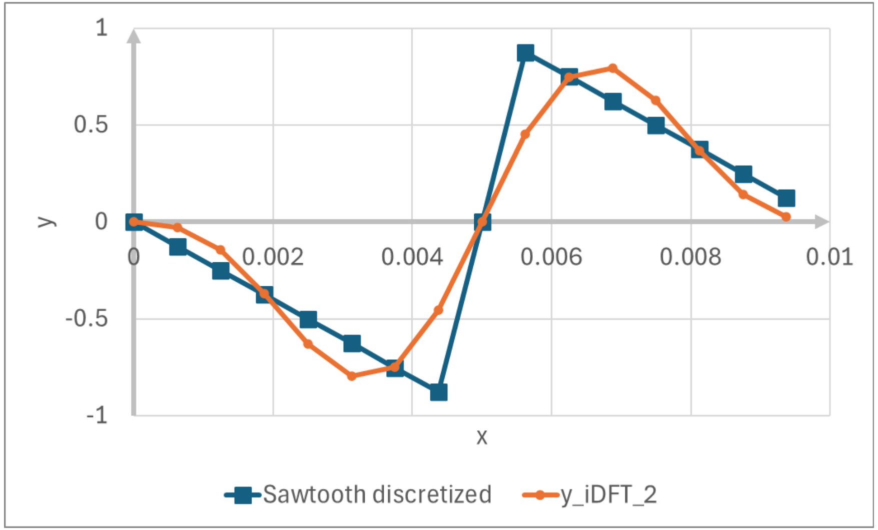
Discrete Fourier Transformation



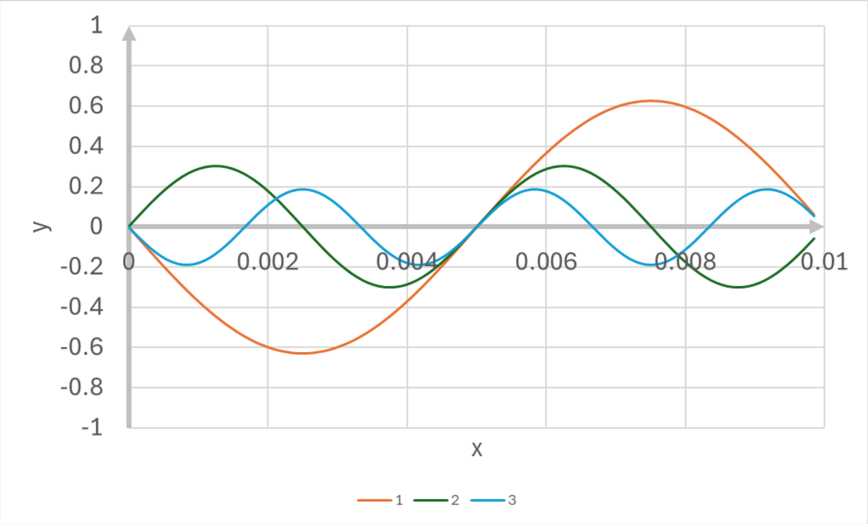


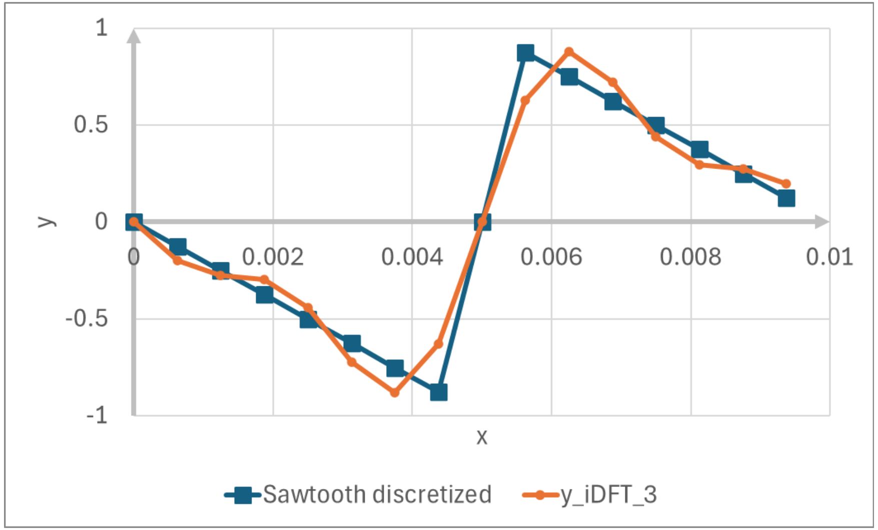
Discrete Fourier Transformation



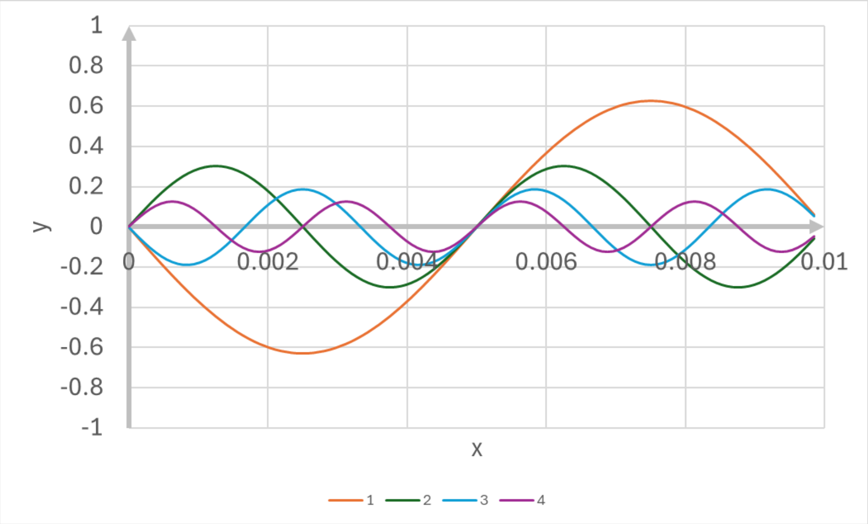


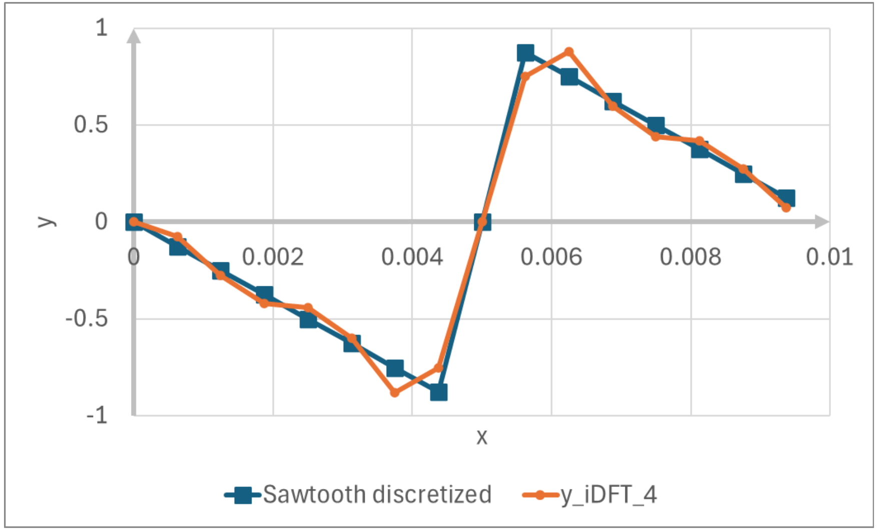
Discrete Fourier Transformation



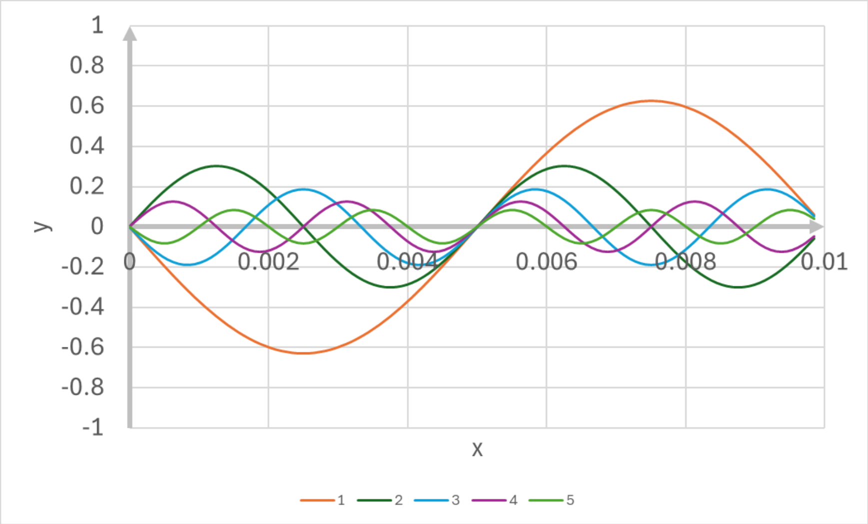


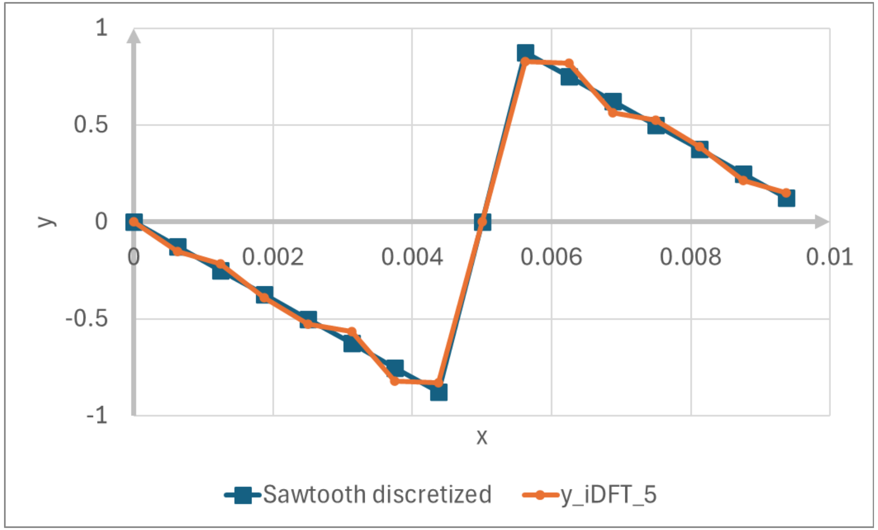
Discrete Fourier Transformation



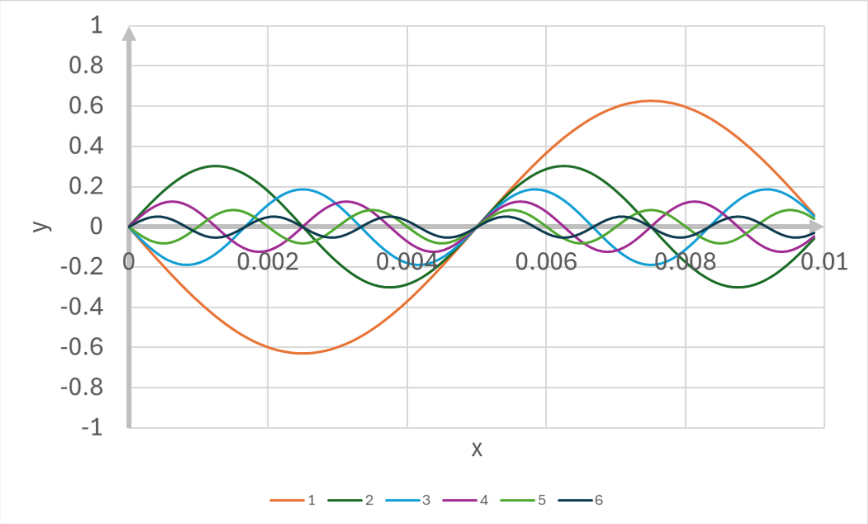


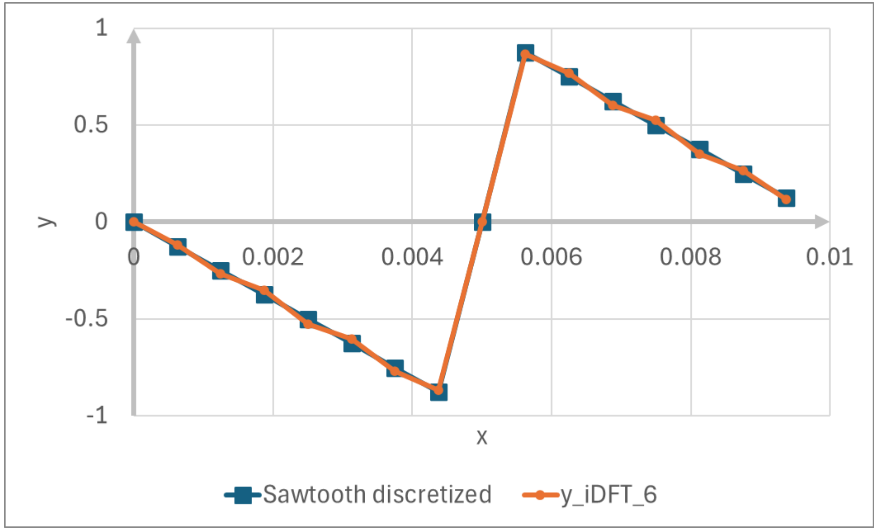
Discrete Fourier Transformation



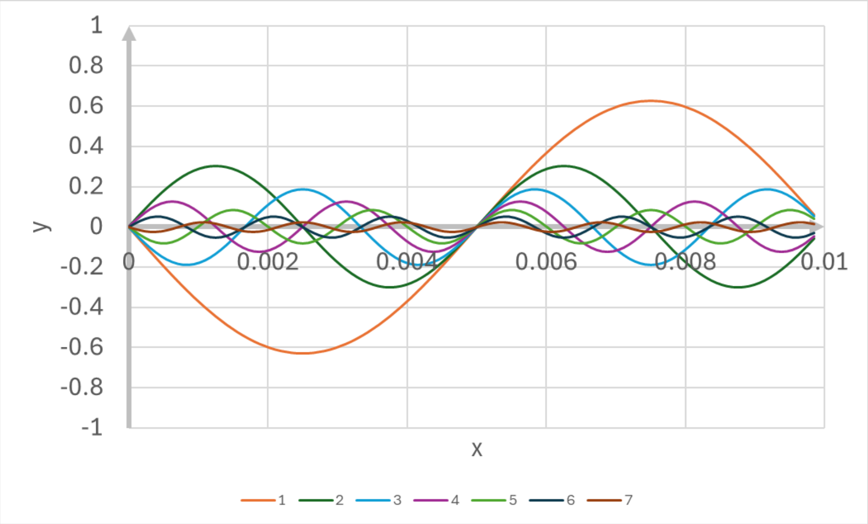


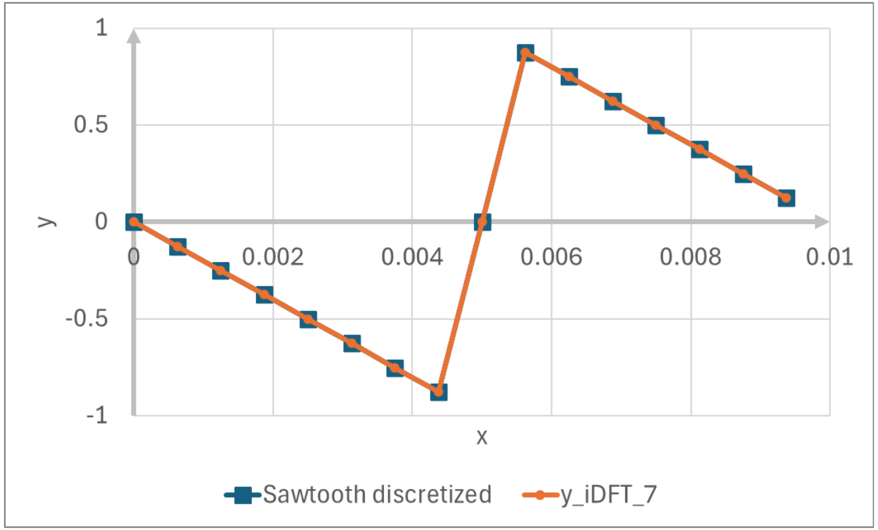
Discrete Fourier Transformation





Discrete Fourier Transformation





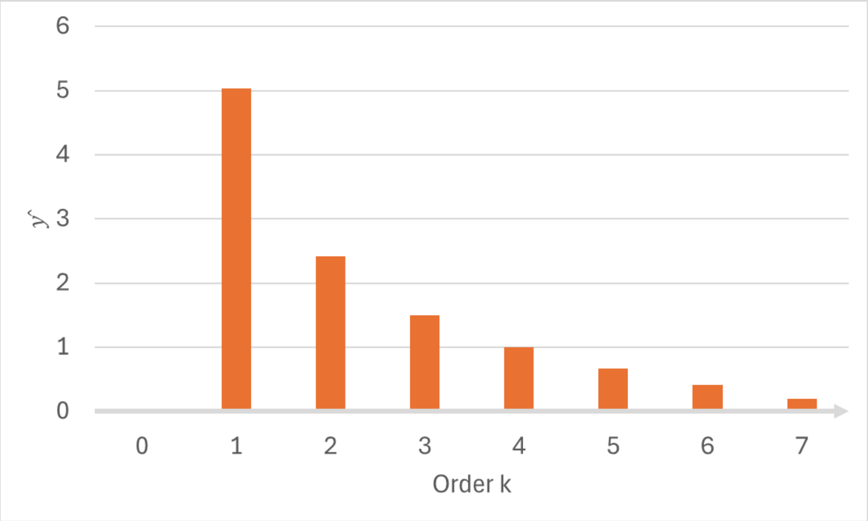
Discrete Fourier Transformation

Fourier Coefficients

- $\hat{y}_k = \sum_{j=0}^{N-i} \mathrm{e}^{-2\pi \mathrm{i}\frac{jk}{N}} y_j$
- for $k = 0, \ldots N-1$

In Matrix Notation

- $\hat{\mathbf{y}} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{y}$
- with $\mathbf{\Phi}[j,k] = \mathrm{e}^{-2\pi \mathrm{i}\frac{jk}{N}}$



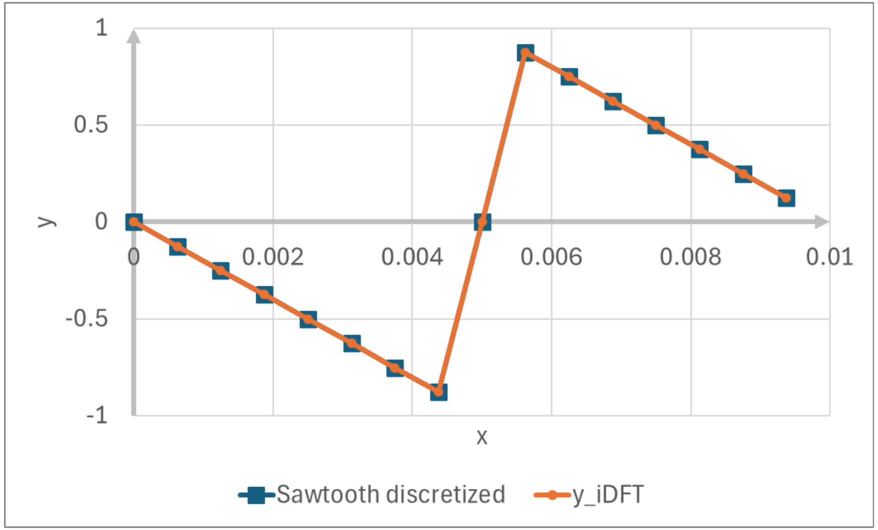
Discrete Fourier Transformation

Inverse Discrete Fourier Transformation

- $y_k = \frac{1}{N} \sum_{j=0}^{N-i} \mathrm{e}^{2\pi \mathrm{i} \frac{jk}{N}} \hat{y}_j$
- for $k = 0, \ldots N-1$

In Matrix Notation

- $\mathbf{y} = [\boldsymbol{\Phi}^{\mathrm{T}}]^{-1}\, \hat{\mathbf{y}}$
- with $\boldsymbol{\Phi}[j,k] = \mathrm{e}^{-2\pi \mathrm{i} \frac{jk}{N}}$



Discrete Fourier Transformation

Link to model order reduction?

Discretized signal

- $y = \{y_0, \dots, y_{N-1}\}$ $y \in \mathbb{R}^N$, Order N

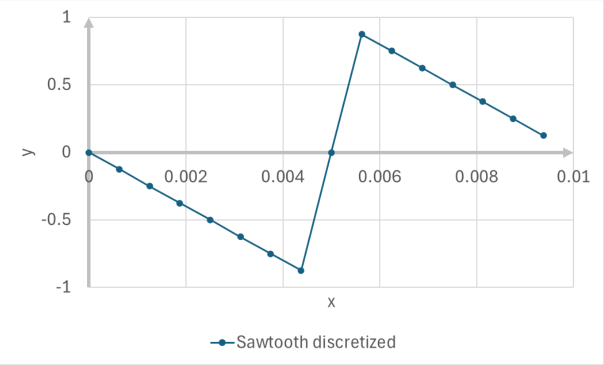
Representation by DFT

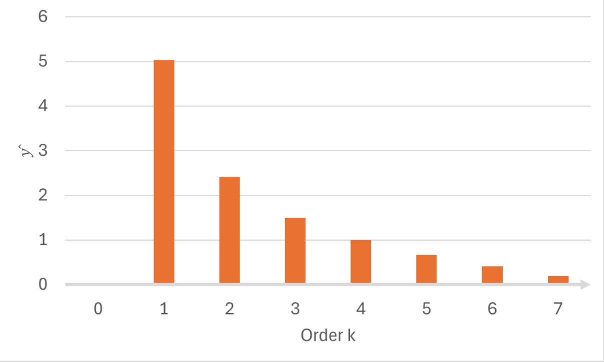
- $\hat{y}_k = \sum_{j=0}^{N-i} \mathrm{e}^{-2\pi i \frac{jk}{N}} y_j$, $\hat{y} \in \mathbb{R}^N$, Order N
- for $k = 0, \dots N-1$

The same data is represented

- …but in different coordinates
- They can be interpretated in a different way

Let's call $\hat{y}$ generalized coordinates



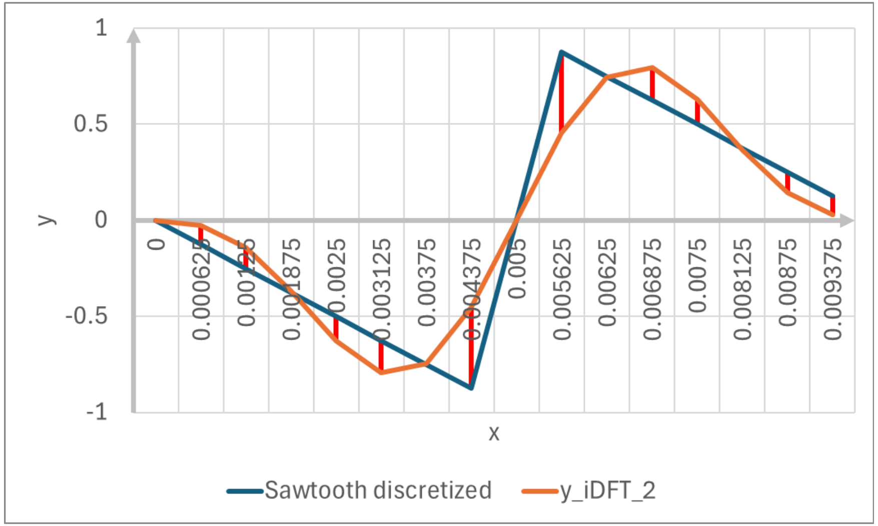


Discrete Fourier Transformation

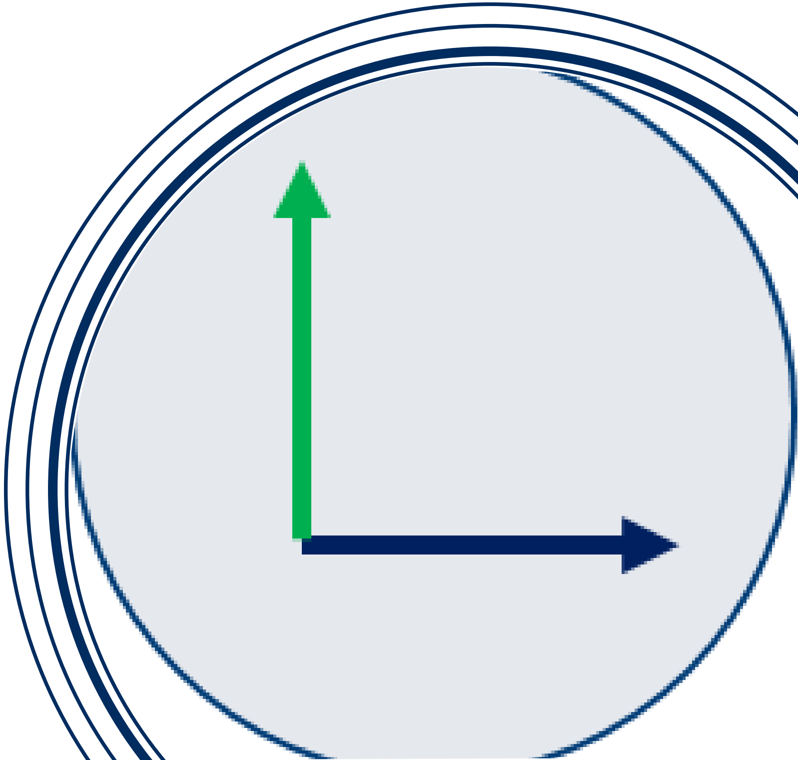
…is some kind of model order reduction

Reduce Dimension

- By taking less Fourier coefficients
- Error is introduced



Vector Space



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Vector Space $\mathbf{V}$

Vector $\mathbf{a}$

Mechanical engineer

- Force

CFD

- Velocity

Electrical engineer

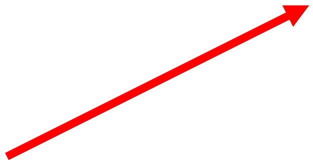
- Voltage in the complex plane

My girlfriend

- There is nothing this arrow points on

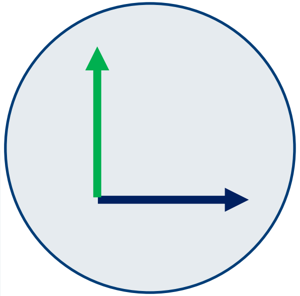
Mathematician

- A vector in $\mathbb{R}^2$



Vector Space $\mathbf{V}$

Basis of Vector Space



Basis $\mathbf{B}$

- …of a vector space $\mathbf{V}$ with basis vectors $\mathbf{b}_1$ to $\mathbf{b}_m$
- $\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_m] \in \mathbb{R}^m$
- m is the dimension of the vector space

Those basis vectors must be linear independent

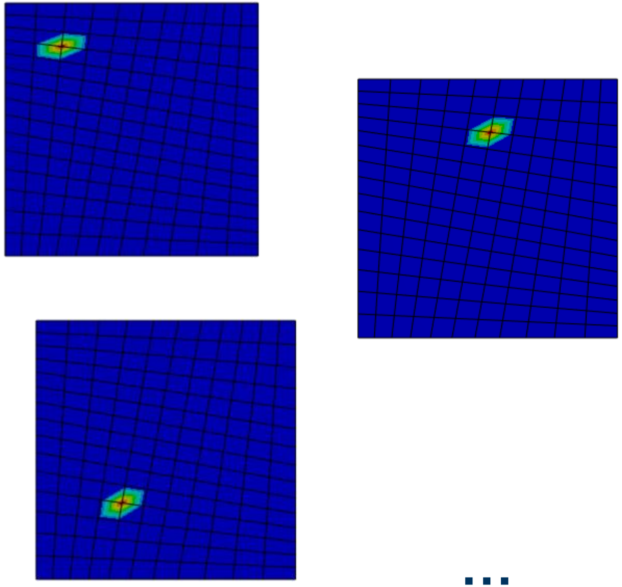
- Each vector can be represented as a linear combination of basis vectors and this representation is unique
- Euclidian basis $\mathbb{R}^m$ (also called standard, natural or canonical basis)
  - $\mathbf{b}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \mathbf{b}_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$
  - $\mathbf{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$

Vector Space V

Basis of Vector Space

FEM

- “1” at one node, all others are 0
- $\in \mathbb{R}^N$
- N is the number of dofs



…

Vector Space V

Coordinates of a vector

Vector a

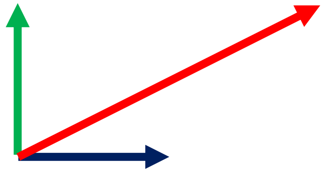
- …with its coordinates a_1 to a_N with respect to basis $\mathbf{B}_1$
- $\mathbf{a} = \{a_1, \dots, a_N\} \quad \in \mathbb{R}^N$
- E.g. $\mathbf{a} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$
- $2\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + 1\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \mathbf{B}_1\mathbf{a} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$

Each vector can be represented

- …as a linear combination of basis vectors
- and this representation is unique

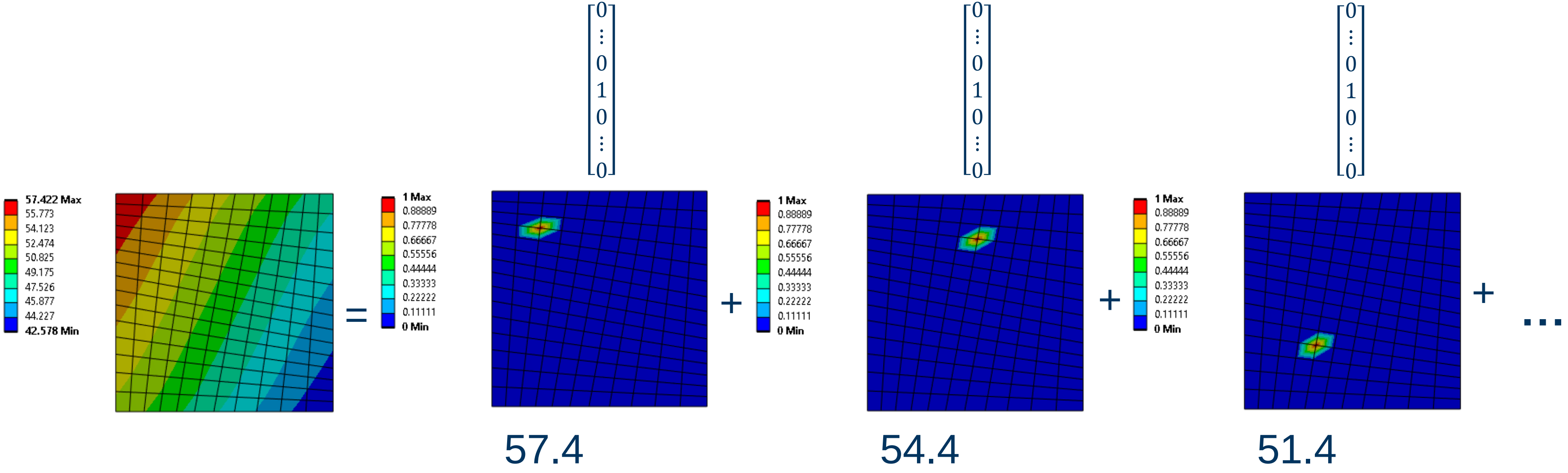
In reduced order modeling

- $\mathbf{Ba}$ is called Expansion



Vector Space V

Coordinates of a vector

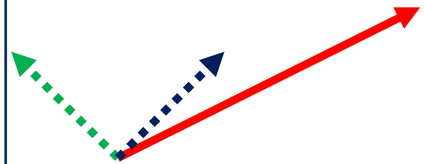


Vector Space V

The basis is not unique!

Different basis

- E.g. $\mathbf{b}_3 = \begin{Bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{Bmatrix}$, $\mathbf{b}_4 = \begin{Bmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{Bmatrix}$
- $\mathbf{B}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$
- spans the same vector space

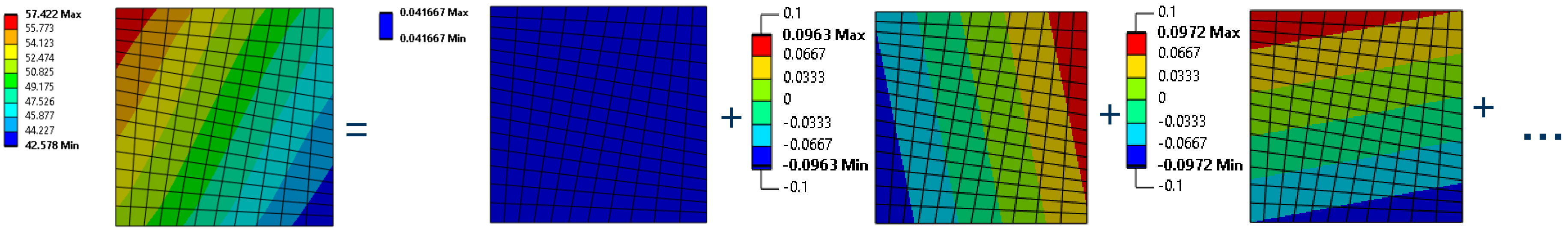


Vector a

- …with its coordinates $\hat{a}_1$ to $\hat{a}_N$ with respect to basis $\mathbf{B}_2$
- $\hat{\boldsymbol{a}} = \begin{Bmatrix} 2.12 \\ -0.71 \end{Bmatrix}$
- $2.12 \begin{Bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{Bmatrix} + -0.71 \begin{Bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{Bmatrix} = \mathbf{B}_2 \hat{\boldsymbol{a}}$

Vector Space V

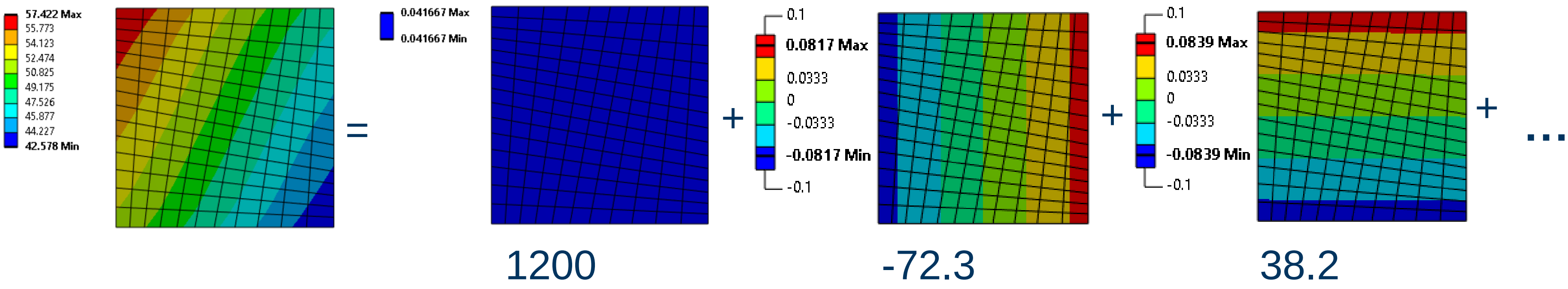
The basis is not unique!



1200 -63.6 51.4

Vector Space V

The basis is not unique!



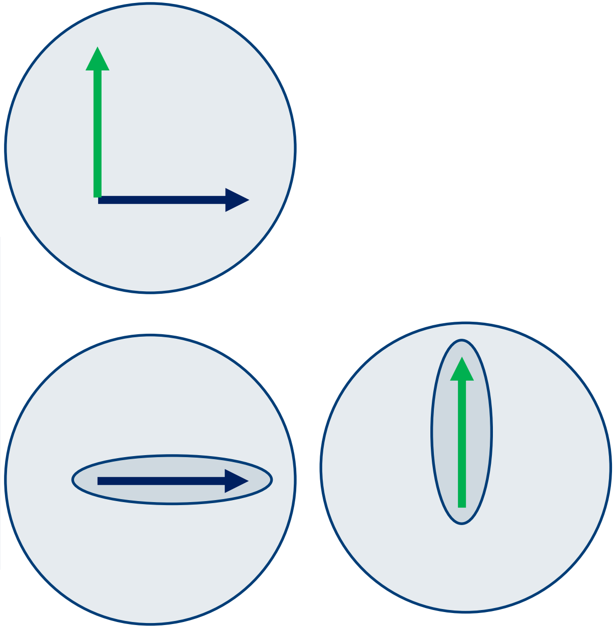
Vector Space $\mathbf{V}$

Truncated Basis

One can truncate the basis by neglecting some basis vectors

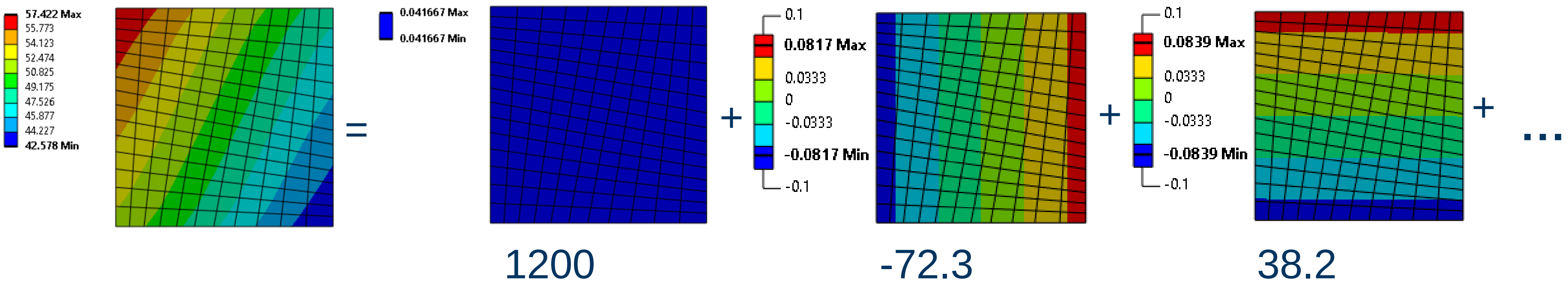
- $\mathbf{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- E.g. $\mathbf{B}_{1_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\mathbf{B}_{1_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The truncated basis spans a subspace of $\mathbf{V}$



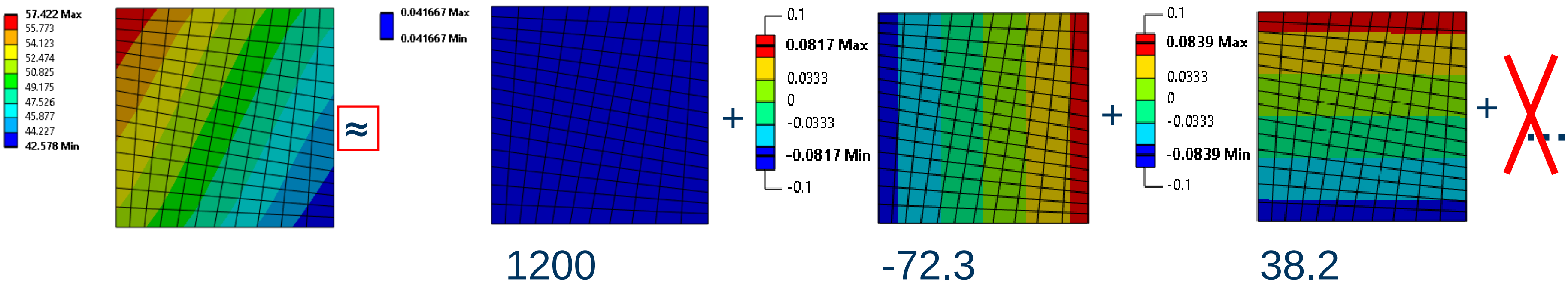
Vector Space V

Truncated Basis



Vector Space V

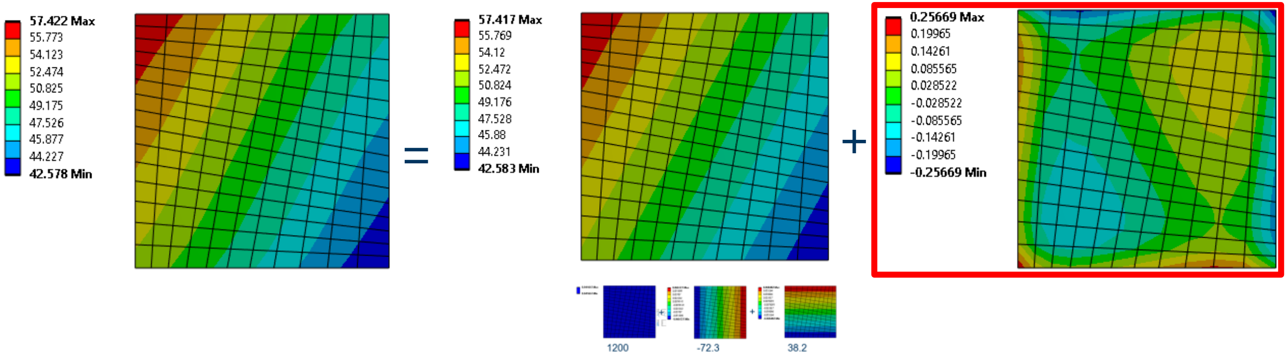
Truncated Basis



Vector Space V

Truncated Basis

Error due to truncated basis

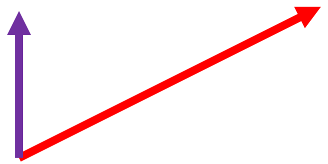


Vector Space V

Scalar product / Inner Product

Scalar Product

- Mathematical relationship that assigns a scalar to two vectors
- Often denoted as by $\langle \mathbf{a}, \mathbf{c} \rangle$
- $\langle \mathbf{a}, \mathbf{c} \rangle = \mathbf{a}^{\mathrm{T}}\mathbf{c} = \sum_{j=1}^{N} a_j c_j = a_1 c_1 + \cdots + a_N c_N$
- E.g. $\mathbf{a} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$, $\mathbf{c} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$
- $\mathbf{a}^{\mathrm{T}}\mathbf{c} = 2 \cdot 0 + 1 \cdot 1 = 1$

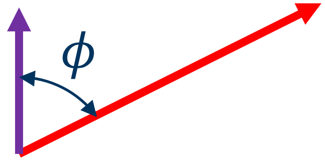


Vector Space V

Scalar product / Inner Product

Geometric interpretation in $\mathbb{R}^2$

- $\mathbf{a}^{\mathrm{T}}\mathbf{c} = \|\mathbf{a}\|\|\mathbf{c}\|\cos(\phi)$
- $\phi = \arccos(\frac{\mathbf{a}^{\mathrm{T}}\mathbf{c}}{\|\mathbf{a}\|\|\mathbf{c}\|})$
- E.g. $\mathbf{a} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$, $\mathbf{c} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$
- $\phi = \arccos\left(\frac{2 \cdot 0 + 1 \cdot 1}{\sqrt{5} \cdot 1}\right) = 63°$

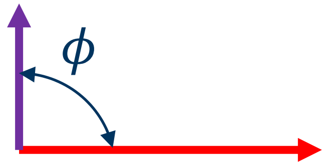


Vector Space $\mathbb{V}$

Orthogonality

Two vectors are orthogonal if the scalar product is 0

- In $\mathbb{R}^2$ the angle is 90°
- E.g. $\mathbf{a} = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$, $\mathbf{c} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$
- $\mathbf{a}^\mathrm{T}\mathbf{c} = 2 \cdot 0 + 0 \cdot 1 = 0$



Vector Space V

Orthogonal Basis

Orthogonal basis

- The scalar product is 0 for $\mathbf{b}_i^{\mathrm{T}}\mathbf{b}_j$ with $i \neq j$

$\mathbf{B}_1^{\mathrm{T}}\mathbf{B}_1$

- $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathbf{B}_2^{\mathrm{T}}\mathbf{B}_2$

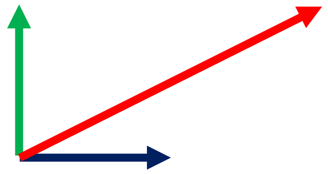
- $= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \cdot \frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \cdot \frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Vector Space V

Orthogonal Projection

Given vector a

- $\mathbf{a} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$

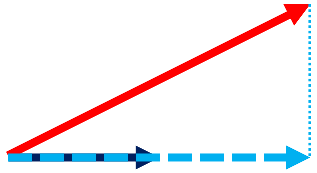


w.r.t to basis $\mathbf{B}_1$

- $\mathbf{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $\mathbf{B}_1\mathbf{a} = 2\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + 1\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$

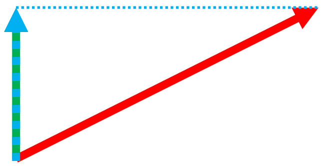
Projection onto truncated basis $\mathbf{B}_{1_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- $\hat{a}_1 = \mathbf{B}_{1_1}^{\mathrm{T}}\mathbf{a} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = 2$



Projection onto truncated basis $\mathbf{B}_{1_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $\hat{a}_2 = \mathbf{B}_{1_2}^{\mathrm{T}}\mathbf{a} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = 1$

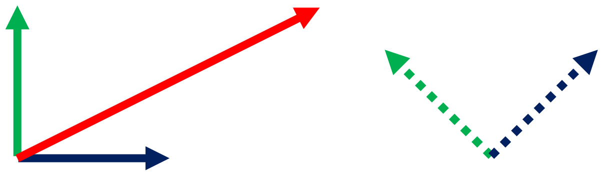


Vector Space V

Change of Basis

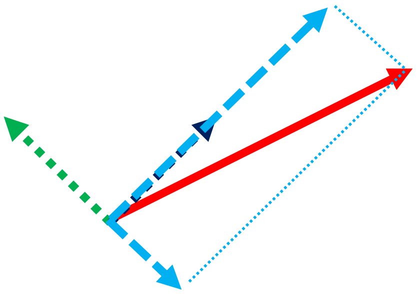
Change of basis of vector a

- $\mathbf{a} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$
- From $\mathbf{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to $\mathbf{B}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$



Linear system of equations

- $\mathbf{B}_1 \mathbf{a} = \mathbf{B}_2 \hat{\mathbf{a}}$



Solution

- $\hat{\mathbf{a}} = \mathbf{B}_2^{-1} \, \mathbf{B}_1 \mathbf{a}$

Vector Space V

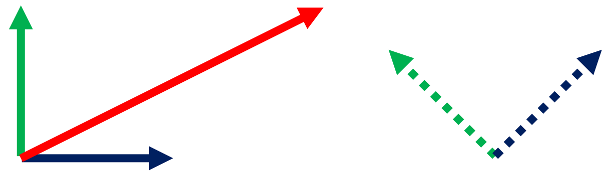
Change of Basis - Why are orthogonal bases that nice?

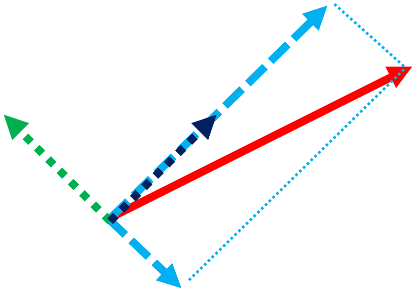
Solution

- $\hat{\mathbf{a}} = \mathbf{B}_2^{-1}\,\mathbf{B}_1\,\mathbf{a}$

Why are orthogonal bases that nice?

- $\mathbf{B}_2^{-1} = \mathbf{B}_2^{\mathrm{T}}$
- $\hat{\mathbf{a}} = \mathbf{B}_2^{\mathrm{T}}\,\mathbf{B}_1\,\mathbf{a}$
- $\hat{\mathbf{a}} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2.12 \\ -.71 \end{Bmatrix}$





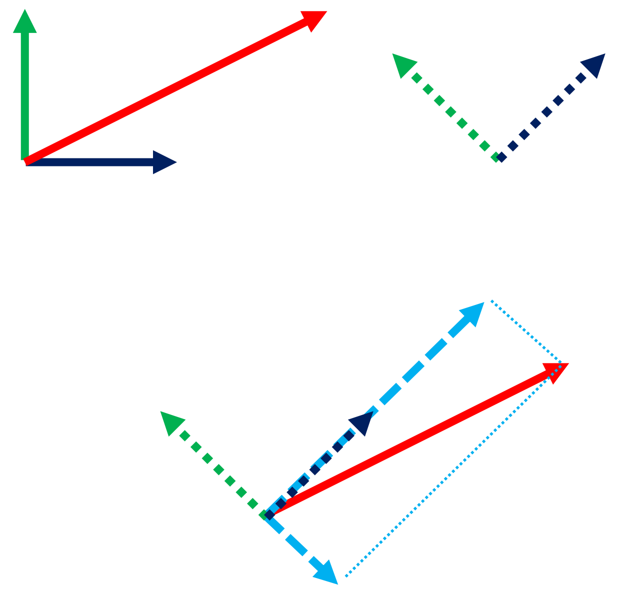
Vector Space V

Change of Basis - Projection

$$\hat{\mathbf{a}} = \mathbf{B}_2^{T}\,\mathbf{B}_1\,\mathbf{a}$$

In reduced order modeling

- If not denoted differently, the basis $\mathbf{B}_1$ is the Euclidian basis
- $\hat{\mathbf{a}} = \mathbf{\Phi}^{T}\,\mathbf{a}$ is just called Projection
- The truncated basis is written as $\mathbf{\Phi}$, describing the subspace
- $\hat{\mathbf{a}}$ are called generalized coordinates



Vector Space V

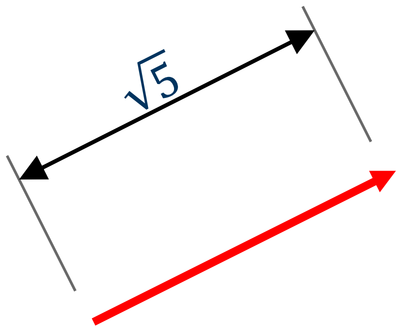
Norm

Norm

- Assigns a non-negative real number to an element of the vector space
- Often denoted as ||·||.
- Generalization of the intuitive notion of "length" in the physical world

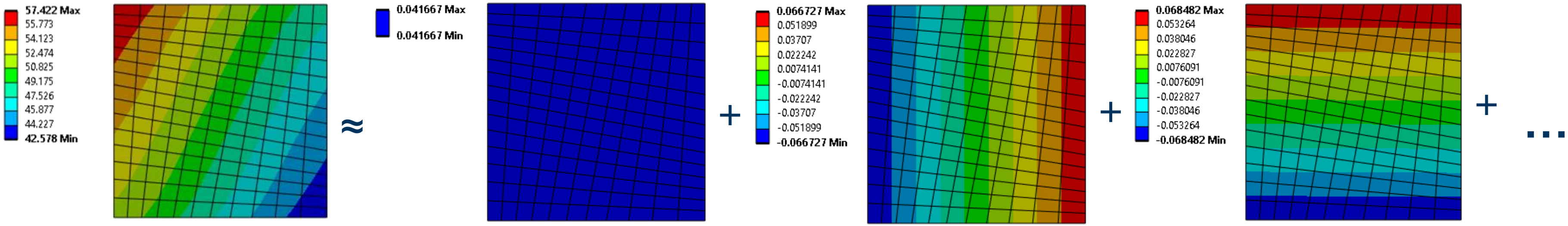
Often the norm is defined by the scalar product

- $\|\mathbf{a}\| = \sqrt{\mathbf{a}^{\mathrm{T}}\mathbf{a}}$
- E.g. $\mathbf{a} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$
- $\|\mathbf{a}\| = \sqrt{2 \cdot 2 + 1 \cdot 1} = \sqrt{5}$
- „L2-Norm“, Euclidian norm



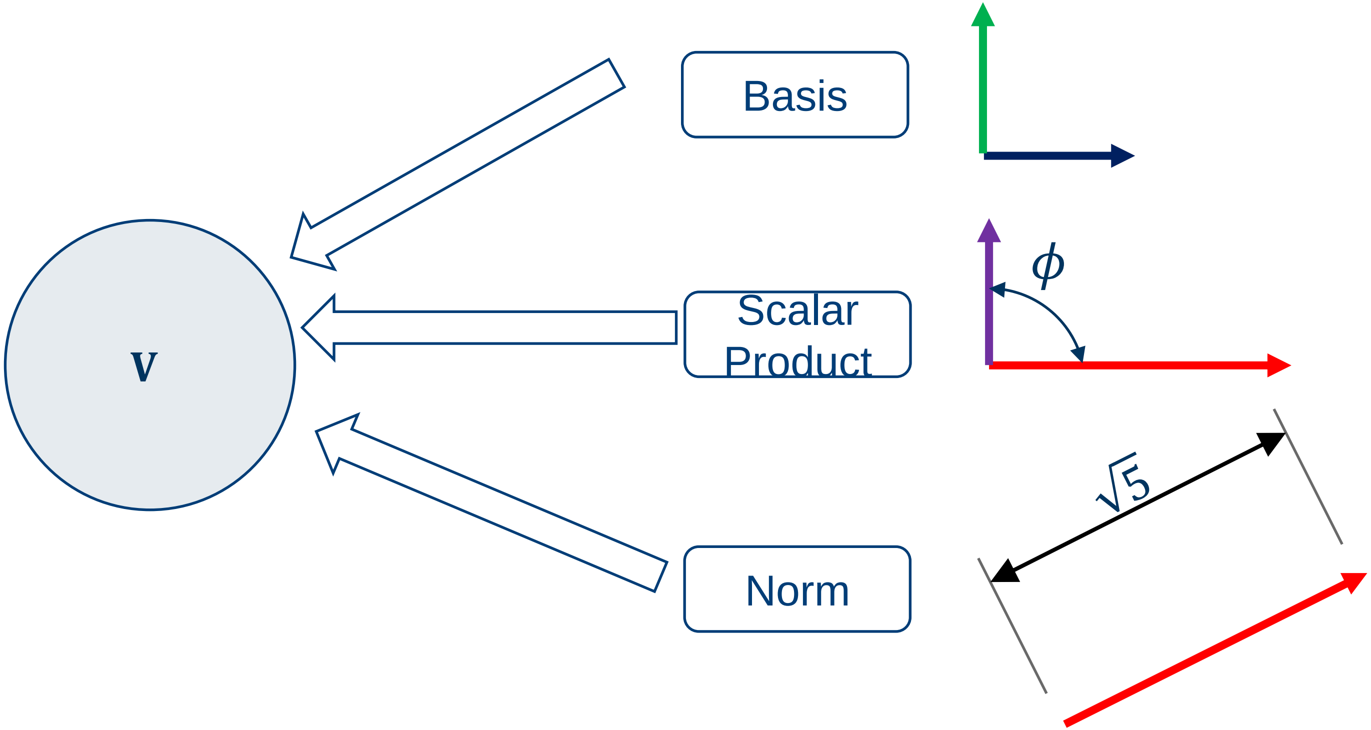
Vector Space V

Normalization of Basis Vectors – Normalized to 1



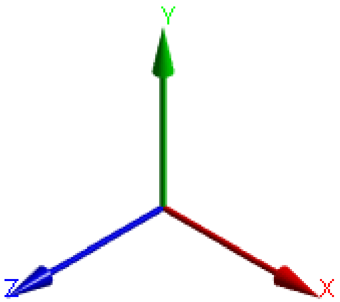
1200   -72.3   38.2

Vector Space V

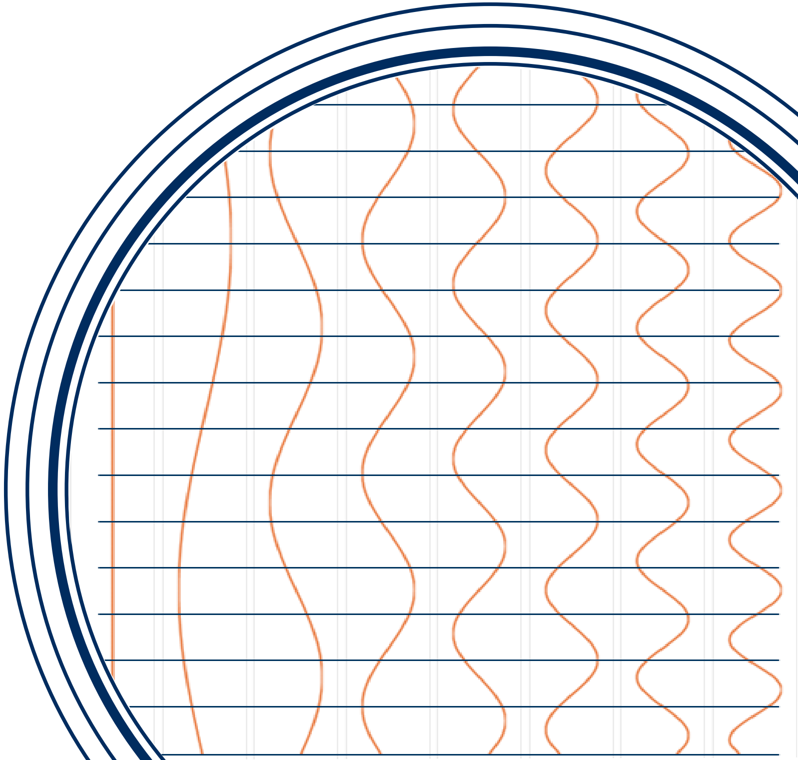


Vector Space V

Engineers…



Discrete Fourier Transformation and Vector Space



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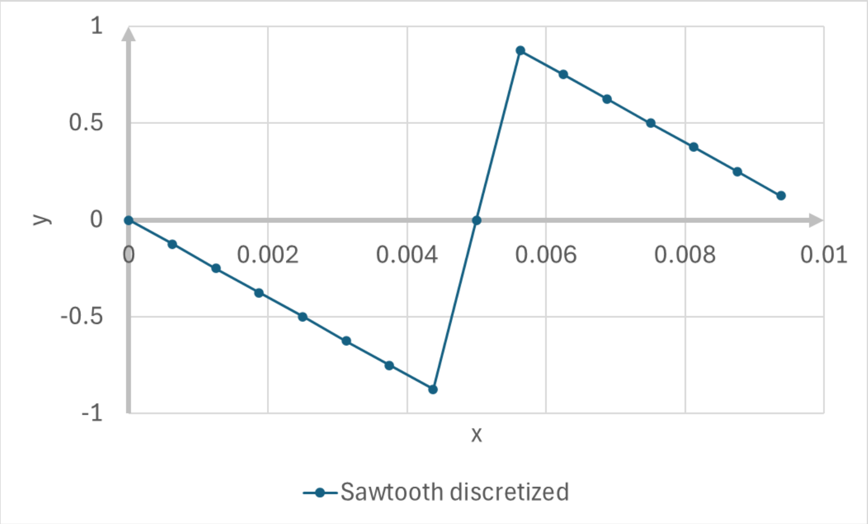
Discrete Fourier Transformation

…is some kind of model order reduction

Discretized

- $\mathbf{y} = \{y_0, \ldots, y_{N-1}\}\ y \in \mathbb{R}^N$
- N Number of timepoints

→ From infinite to N ☺



Discrete Fourier Transformation

…is some kind of model order reduction

- Basis that we have intuitively assumed
- **I**

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
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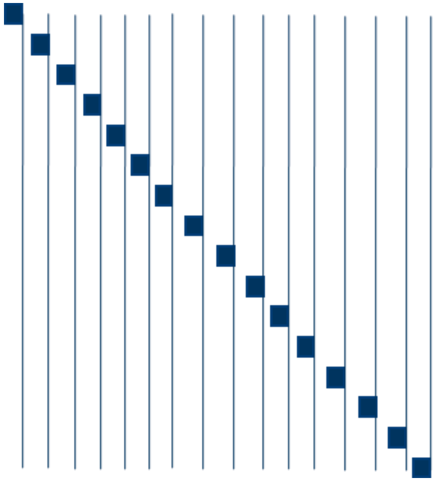
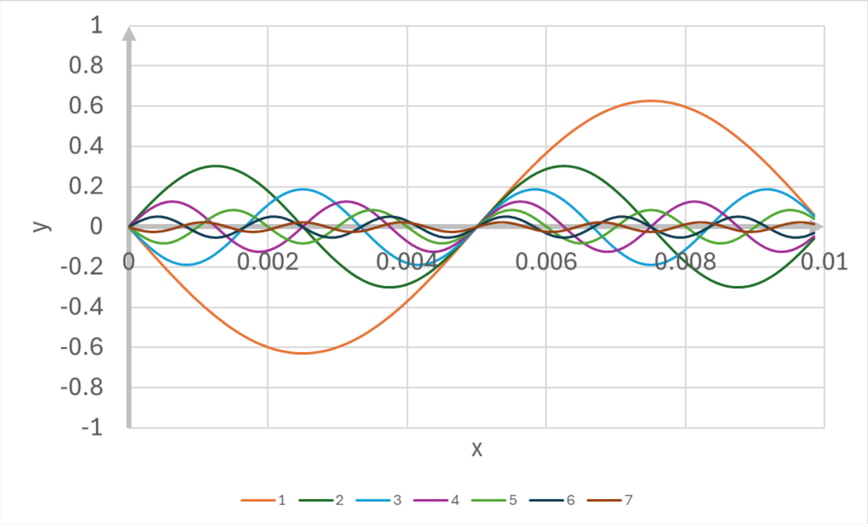
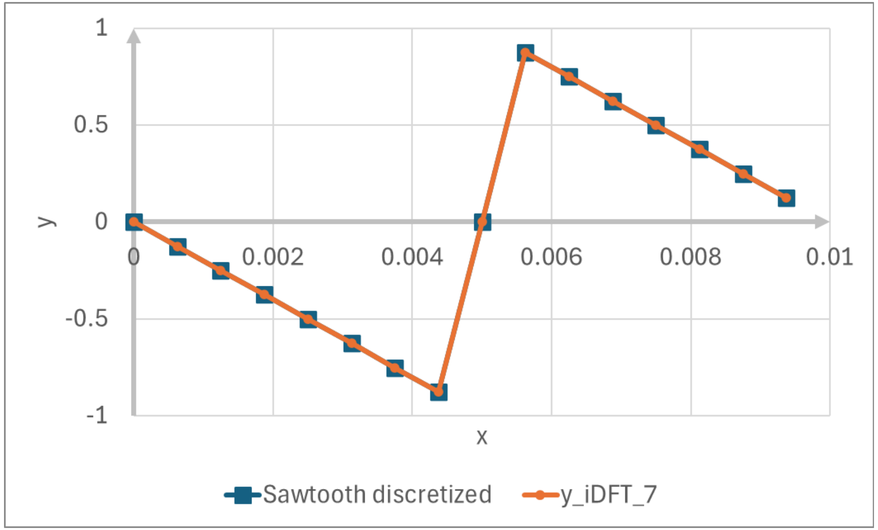
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1



Discrete Fourier Transformation

…is some kind of model order reduction





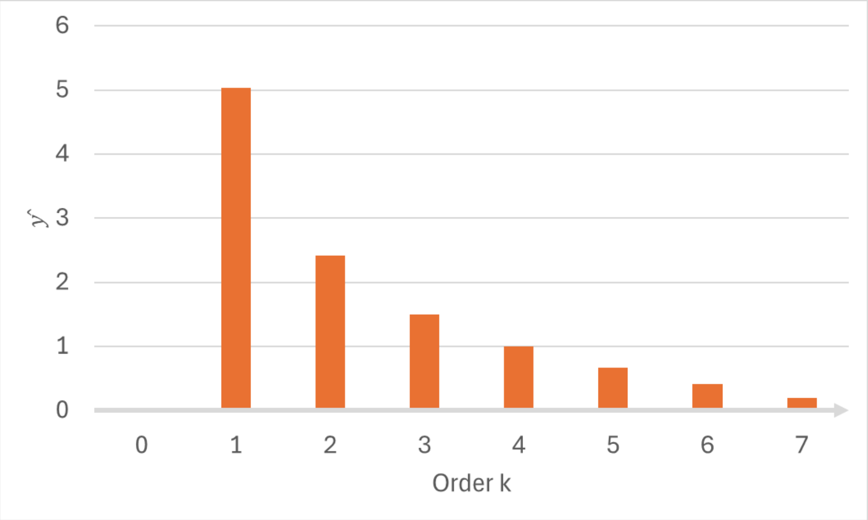
Discrete Fourier Transformation

…is some kind of model order reduction

Discrete Fourier Transformation

- In matrix notation
  - $\hat{\mathbf{y}} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{y}$

Projection!



Discrete Fourier Transformation

…is some kind of model order reduction

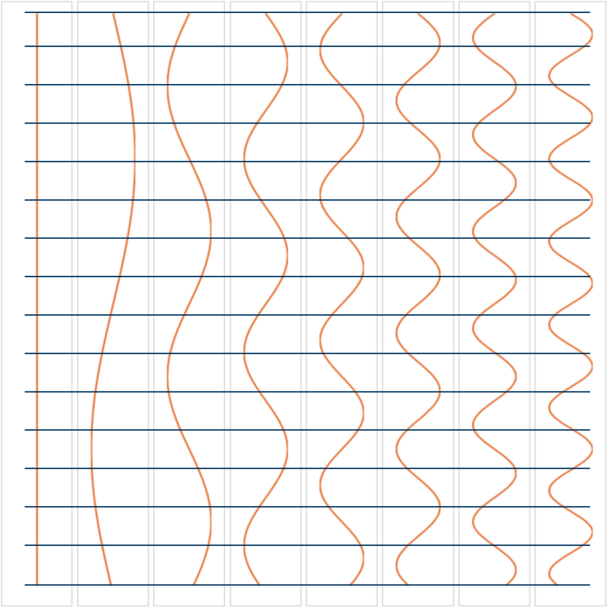
Discrete Fourier Transformation

- In matrix notation
  - $\hat{\mathbf{y}} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{y}$

Projection!

- **Change of basis**

$\mathbf{\Phi}$



Discrete Fourier Transformation

…is some kind of model order reduction

The basis is orthonormal ☺

$\Phi^T \Phi$

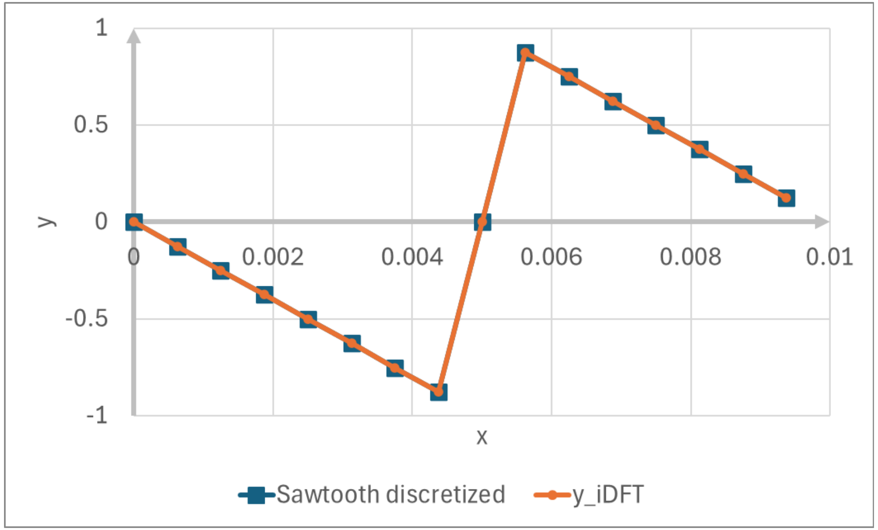
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Discrete Fourier Transformation

…is some kind of model order reduction

Inverse Discrete Fourier Transformation

- In matrix notation
  - $\mathbf{y} = [\mathbf{\Phi}^{\mathrm{T}}]^{-1}\hat{\mathbf{y}}$
  - $[\mathbf{\Phi}^{\mathrm{T}}]^{-1} = [\mathbf{\Phi}^{\mathrm{T}}]^{\mathrm{T}} = \mathbf{\Phi}$
  - $\mathbf{y} = \mathbf{\Phi}\hat{\mathbf{y}}$



Expansion!

Discrete Fourier Transformation

…is some kind of model order reduction

Possibility to reduce the dimension

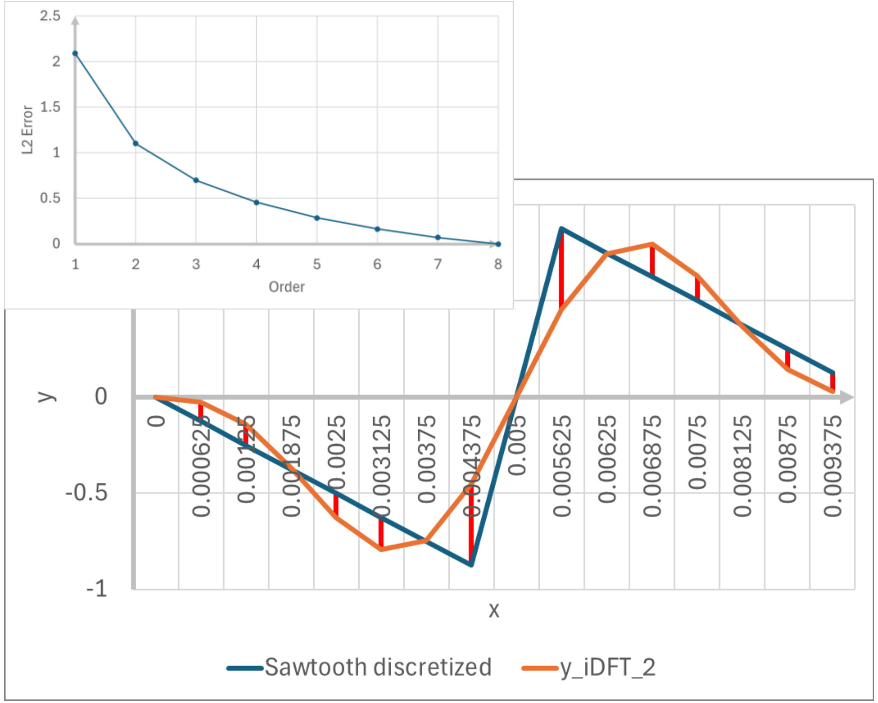
- by taking less Fourier coefficients

Projection

- to a truncated subspace

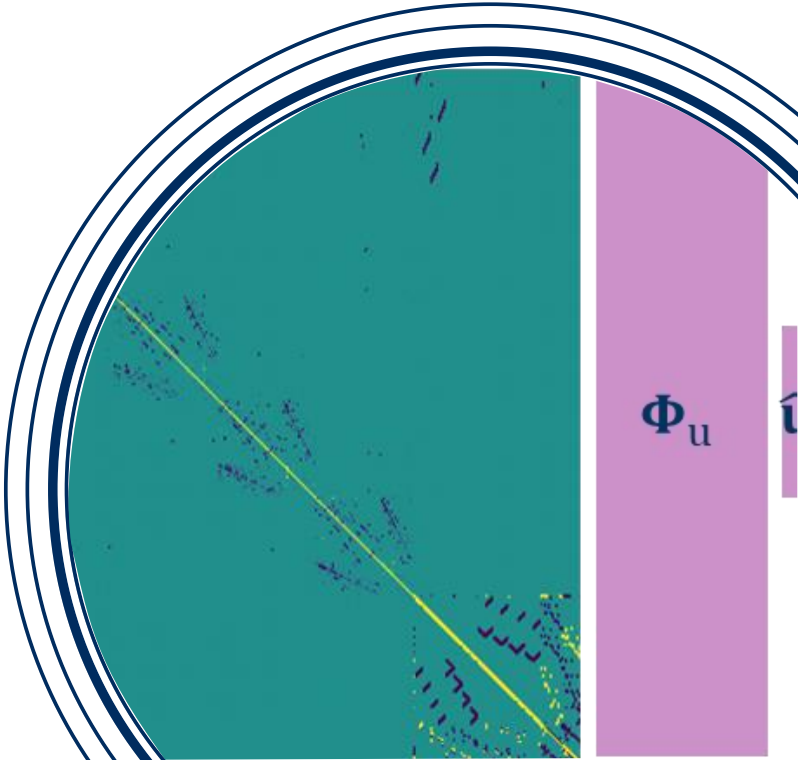
Error

- often measured by the L2-Norm of the difference



Link to Model Order Reduction

PCB Model



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PCB Model

Discretization

Structural model

Linear elements

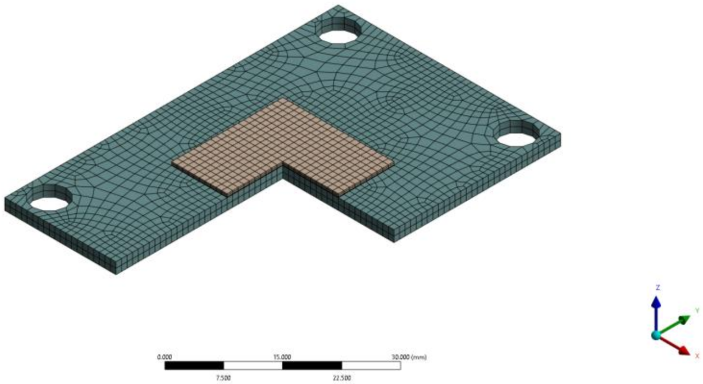
- 7383 nodes
- 5104 elements

3 Degrees of freedom per node

- u_x, u_y, u_z

Displacement vector

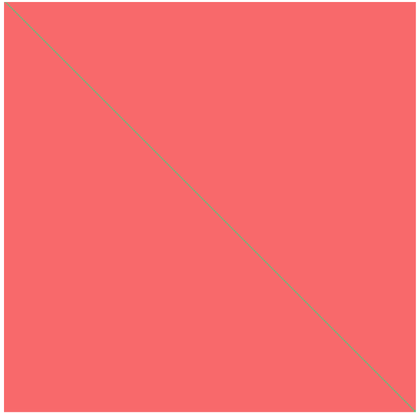
- $\mathbf{u} = \begin{Bmatrix} u_{x_1} \\ u_{y_1} \\ u_{z_1} \\ \vdots \\ u_{y_N} \\ u_{z_N} \end{Bmatrix}$



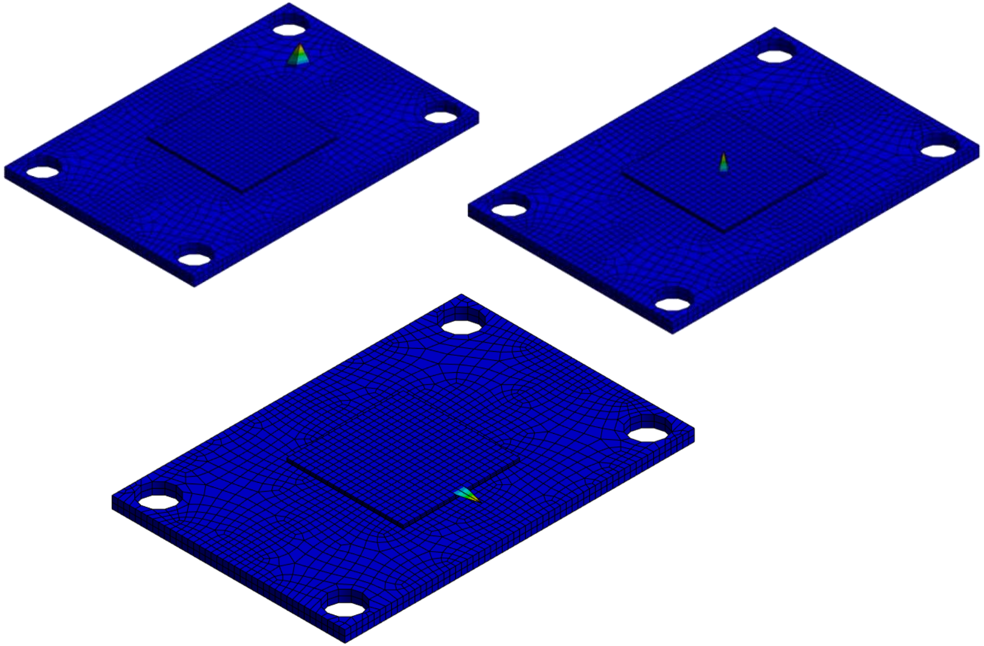
PCB Model

Vector Space

- Vector space of dimension
  - $N = 3 \cdot 7383 = 22149$
  - Euclidian basis
    - $\mathbf{I} \in \mathbb{R}^{N \times N}$



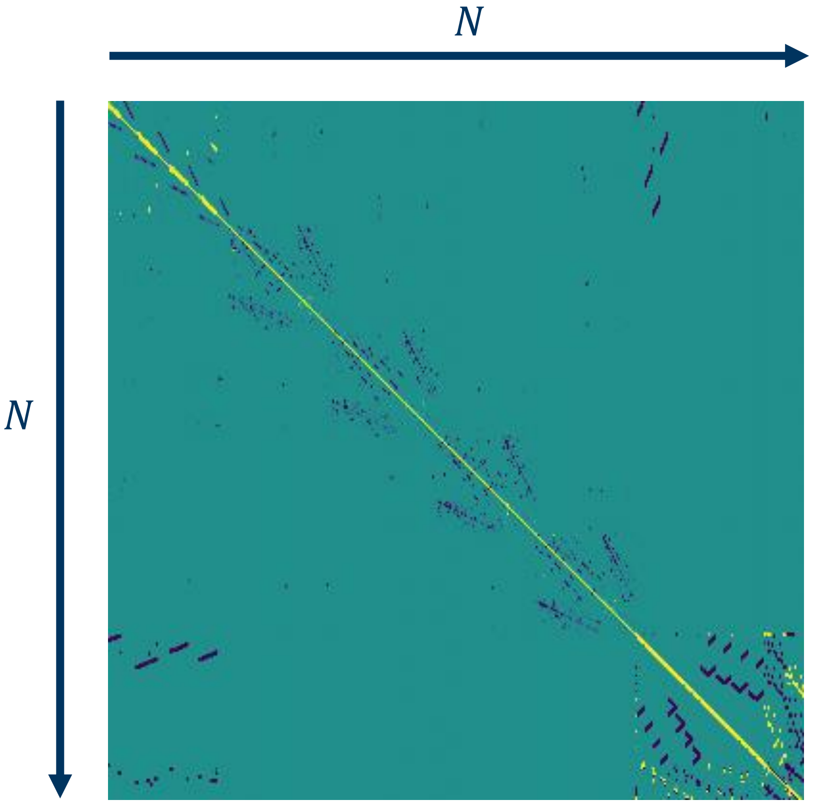
- Exemplary base vectors



PCB Model

Stiffness Matrix $\mathbf{K}$

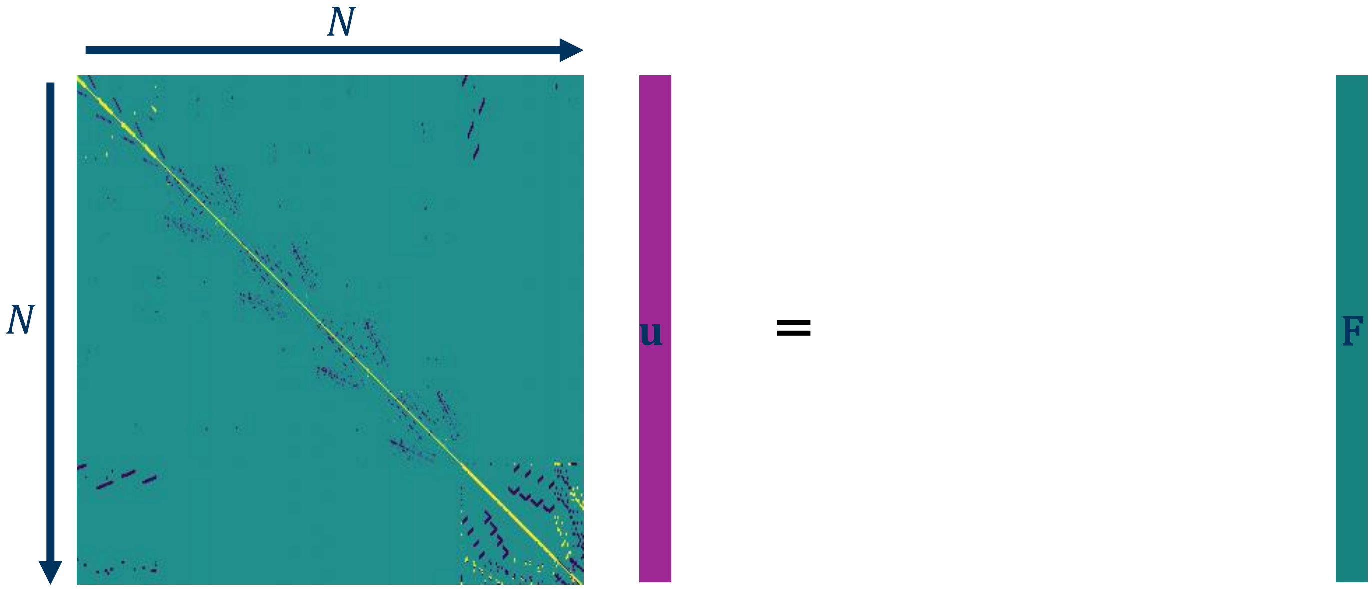
- Symmetric
- Positive definite
- Sparse



PCB Model

Describing Equations

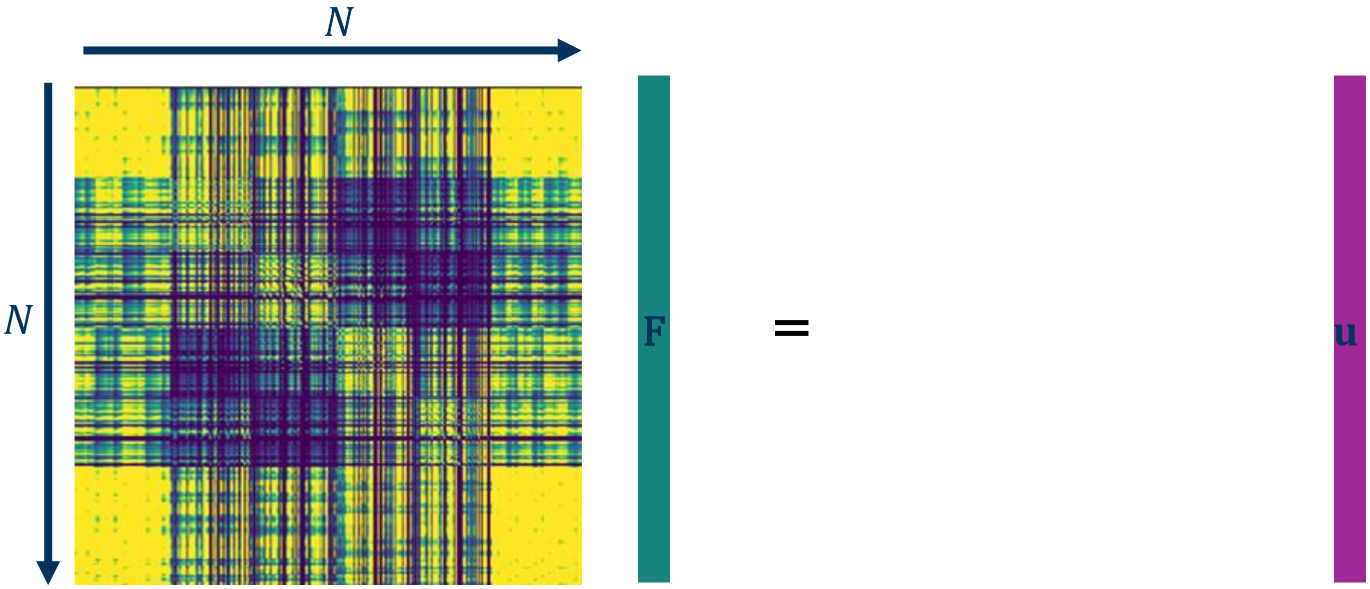
$\mathbf{Ku} = \mathbf{F}$,with external force vector $\mathbf{F}$



PCB Model

Solving

$$\mathbf{K}^{-1}\mathbf{F} = \mathbf{u}$$



PCB Model

Projection of Displacements

Project the displacements

- onto a smaller subspace of dimension $n \ll N$

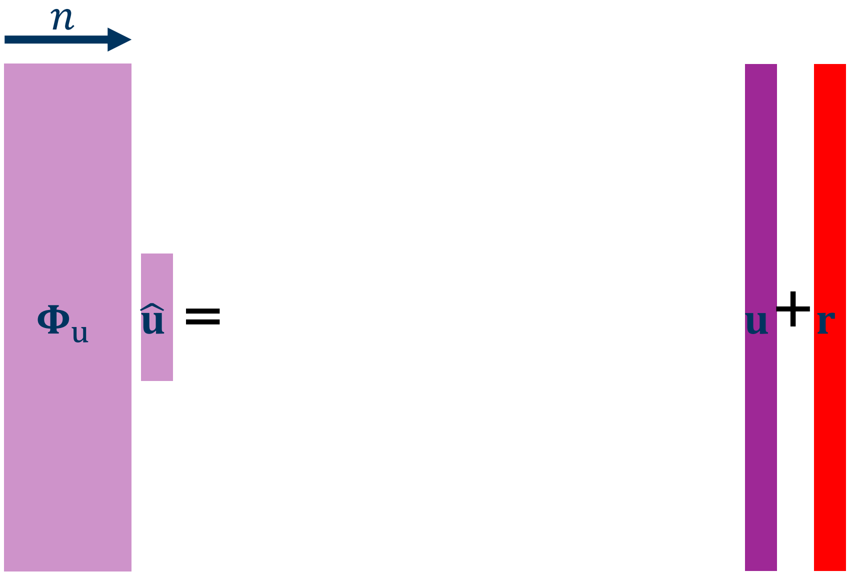
$$\mathbf{u} = \mathbf{\Phi}_\mathrm{u}\hat{\mathbf{u}} + \mathbf{r}$$

- As it is only an approximation, the residual $\mathbf{r}$ is introduced

$$\mathbf{K}\mathbf{u} = \mathbf{F}$$

- $\mathbf{K}\mathbf{\Phi}_\mathrm{u}(\hat{\mathbf{u}} + \mathbf{r}) = \mathbf{F} - \mathbf{K}\mathbf{\Phi}_\mathrm{u}\mathbf{r}$
- $\mathbf{K}\mathbf{\Phi}_\mathrm{u}\hat{\mathbf{u}} = \mathbf{F} - \mathbf{K}\mathbf{\Phi}_\mathrm{u}\mathbf{r}$

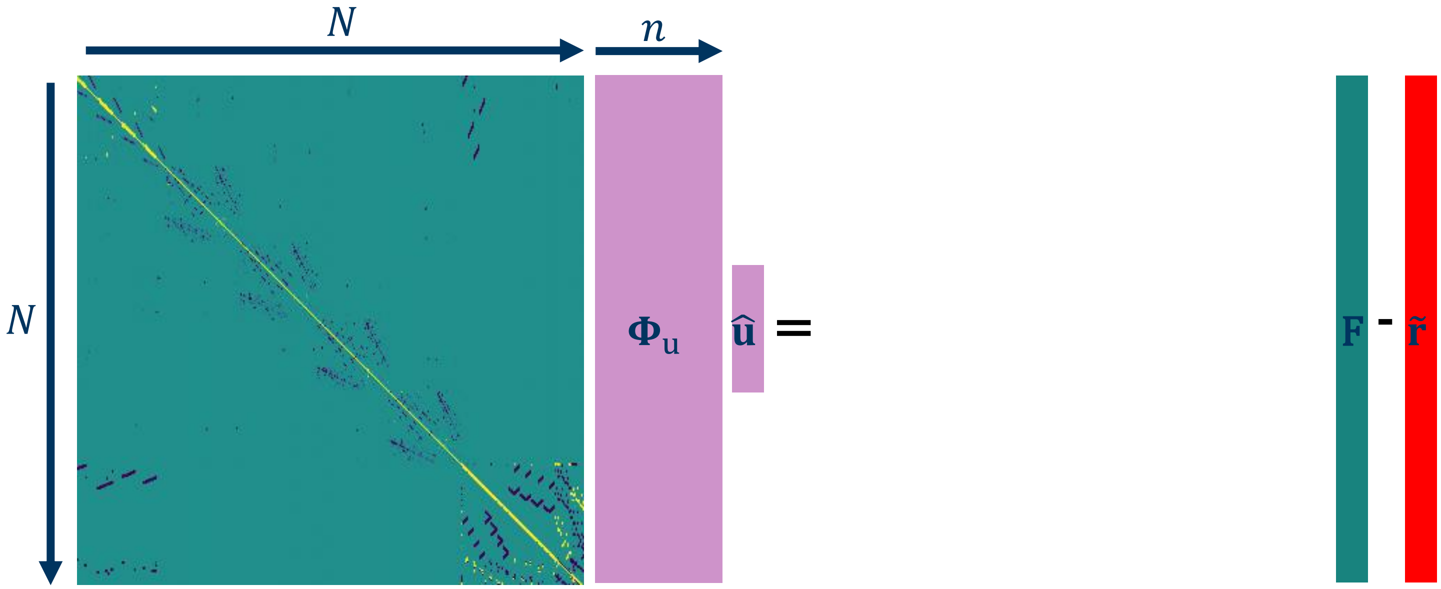
$$\mathbf{K}\mathbf{\Phi}_\mathrm{u}\hat{\mathbf{u}} = \mathbf{F} - \tilde{\mathbf{r}}$$



PCB Model

Projection of Displacements

$$\mathbf{K}\boldsymbol{\Phi}_{\mathrm{u}}\hat{\mathbf{u}} = \mathbf{F} - \tilde{\mathbf{r}}$$



PCB Model

Galerkin Condition

$$\mathbf{K}\boldsymbol{\Phi}_\mathbf{u}\hat{\mathbf{u}} = \mathbf{F} - \tilde{\mathbf{r}}$$

Galerkin condition

- The residual $\tilde{\mathbf{r}}$ is kept orthogonal to the subspace $\boldsymbol{\Phi}_\mathbf{u}$

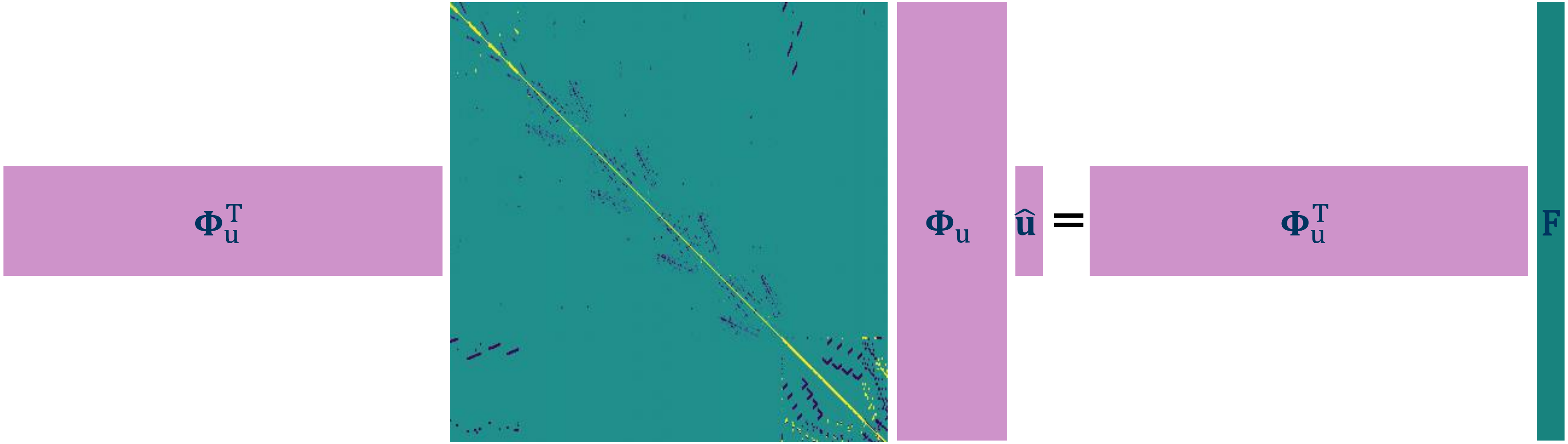
$$\boldsymbol{\Phi}_\mathbf{u}^\mathrm{T}\tilde{\mathbf{r}} = \mathbf{0}$$

- Multiply by $\boldsymbol{\Phi}_\mathbf{u}^\mathrm{T}$
- $\boldsymbol{\Phi}_\mathbf{u}^\mathrm{T}\,\mathbf{K}\boldsymbol{\Phi}_\mathbf{u}\hat{\mathbf{u}} = \boldsymbol{\Phi}_\mathbf{u}^\mathrm{T}\mathbf{F} - \boldsymbol{\Phi}_\mathbf{u}^\mathrm{T}\tilde{\mathbf{r}}$
- $\underbrace{\boldsymbol{\Phi}_\mathbf{u}^\mathrm{T}\,\mathbf{K}\boldsymbol{\Phi}_\mathbf{u}}_{\hat{\mathbf{K}}}\,\hat{\mathbf{u}} = \underbrace{\boldsymbol{\Phi}_\mathbf{u}^\mathrm{T}\mathbf{F}}_{\hat{\mathbf{F}}}$

PCB Model

Reduced Order Model

- $\boldsymbol{\Phi}_u^T \mathbf{K} \boldsymbol{\Phi}_u \hat{\mathbf{u}} = \boldsymbol{\Phi}_u^T \mathbf{F}$



PCB Model

Reduced Order Model

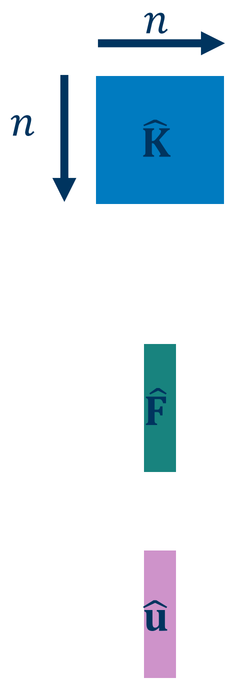
- $\underbrace{\mathbf{\Phi}_\mathbf{u}^\mathrm{T}\,\mathbf{K}\mathbf{\Phi}_\mathbf{u}}_{\hat{\mathbf{K}}}\,\hat{\mathbf{u}} = \underbrace{\mathbf{\Phi}_\mathbf{u}^\mathrm{T}\mathbf{F}}_{\hat{\mathbf{F}}}$

PCB Model

Reduced Order Model

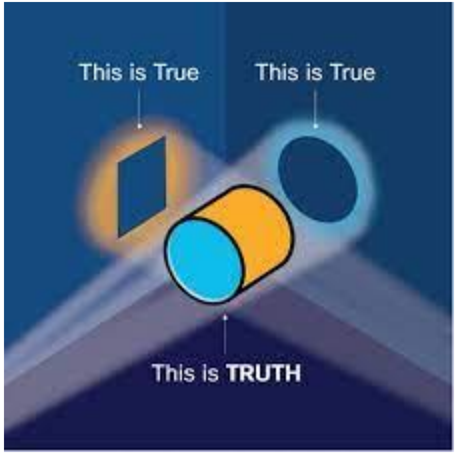
Generalized…

- stiffness matrix $\hat{\mathbf{K}}$
- force vector $\hat{\mathbf{F}}$
- displacement vector $\hat{\mathbf{u}}$



PCB Model

Idea of Projection Based Model Order Reduction



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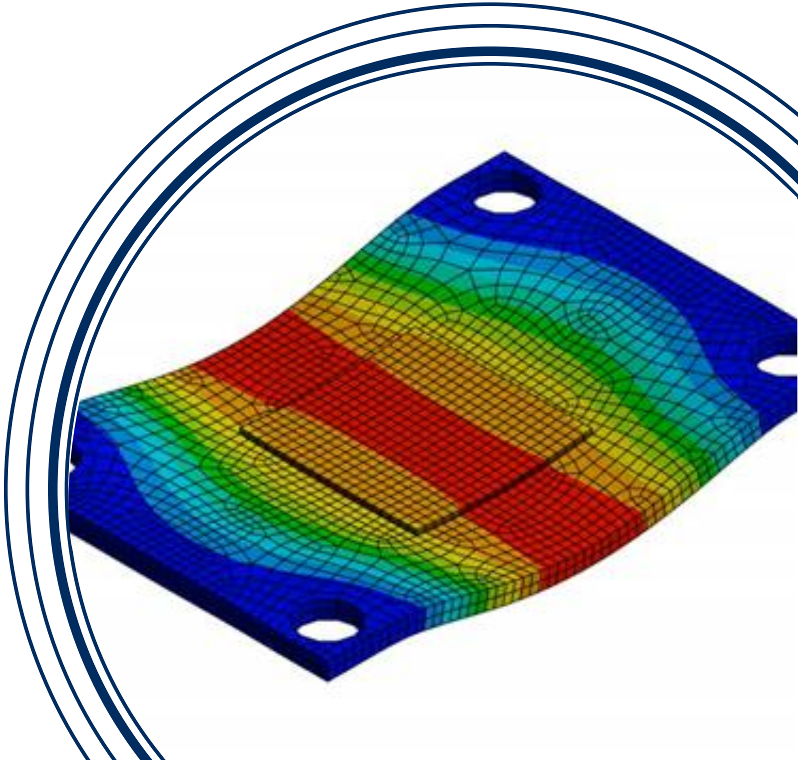
How to find a good Subspace

Methods to Determine Subspaces

Based on System Matrices	Based on Simulation Results
• Modal Subspace • Krylov Subspace	• Gram Schmidt Orthogonalization • Proper Orthogonal Decomposition/ Principal Component Analysis/ Method of Snapshots

How to find a good Subspace

Modal Analysis



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Modal Analysis

"What the system wants to do" – Free Vibrations

Equation of motion

- $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}_{\text{ext}}$
- Damping is neglected
- Independent of external force

→ $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}$

- Displacement vector $\mathbf{u} \in \mathbb{R}^N$
- Velocity vector $\dot{\mathbf{u}} \in \mathbb{R}^N$
- Acceleration vector $\ddot{\mathbf{u}} \in \mathbb{R}^N$
- External Force $\mathbf{F}_{\text{ext}} \in \mathbb{R}^N$

- Mass Matrix $\mathbf{M} \in \mathbb{R}^{N \times N}$
- Stiffness Matrix $\mathbf{K} \in \mathbb{R}^{N \times N}$
- Damping Matrix $\mathbf{C} \in \mathbb{R}^{N \times N}$

Modal Analysis

“What the system wants to do” – Free Vibrations

Equation of motion

- $\mathbf{M\ddot{u}} + \mathbf{C\dot{u}} + \mathbf{Ku} = \mathbf{F}_{\text{ext}}$
- Damping is neglected
- Independent of external force

$$\mathbf{M\ddot{u}} + \mathbf{Ku} = \mathbf{0}$$

For a linear system, free vibrations are harmonic

- $\mathbf{u} = \mathbf{\Phi}_i \cos\omega_i t$
- Decomposition of i-1-DOF-Oscillators

- i-th modeshape $\mathbf{\Phi}_i$
- i-th natural frequency ω_i
- Time t

Modal Analysis

“What the system wants to do” – Free Vibrations

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}$$

- with $\ddot{\mathbf{u}} = -\omega^2\mathbf{u}$

$$(-\omega^2\mathbf{M} + \mathbf{K})\boldsymbol{\phi}_i = \mathbf{0}$$

- This equation is satisfied if
  - $\boldsymbol{\phi}_i = \mathbf{0}$
    - trivial, not of interest
  - Determinant of $(-\omega^2\mathbf{M} + \mathbf{K})$ is $\mathbf{0}$

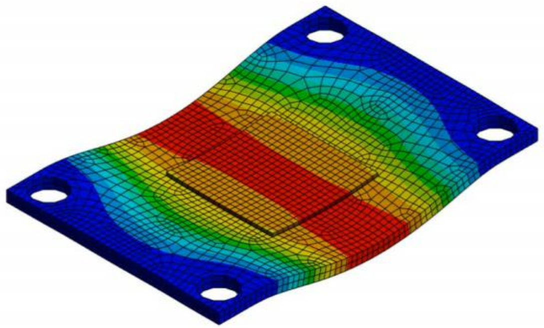
(generalized) eigenvalue problem

- Outputs
  - n mode shapes $\boldsymbol{\Phi} = [\boldsymbol{\Phi}_1 \quad \dots \quad \boldsymbol{\Phi}_n]$
  - n natural frequencies $\boldsymbol{\omega} = \{\omega_1 \quad \dots \quad \omega_n\}$

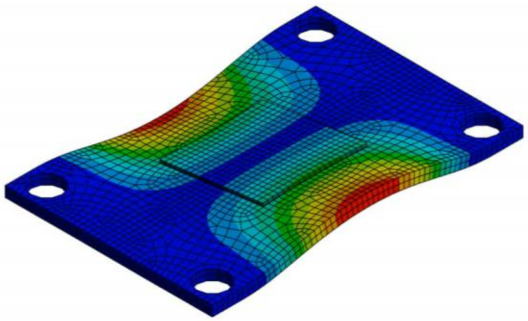
Modal Analysis

“What the system wants to do” – Free Vibrations

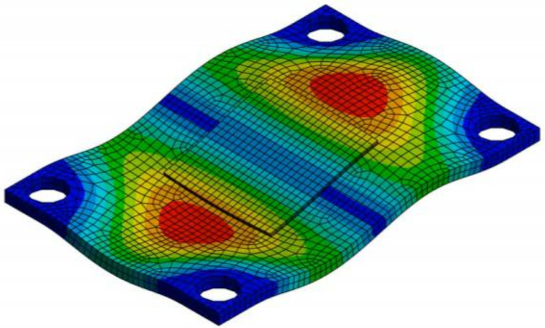
Mode 1, $f_1 = 3.1\mathrm{kHz}$



Mode 2, $f_2 = 4.3\mathrm{kHz}$



Mode 3, $f_3 = 7.3\mathrm{kHz}$



Modal Analysis

"What the system wants to do" – Free Vibrations

Implemented in all commercial FE-Codes

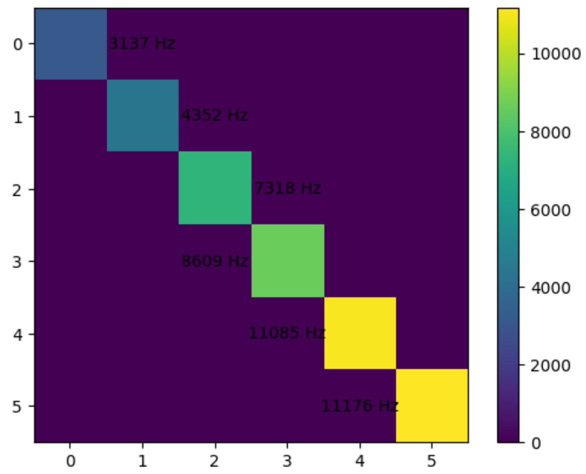
Number of mode shapes?

- The natural frequencies are ordered, from low to high frequencies
- In general: Dimension of the vector space of the model
  - → The same vector space is spanned
- Rule of thumb:
  - Consider mode shapes up to twice the excitation frequency
    - → Modal Truncation

Properties of mode shapes

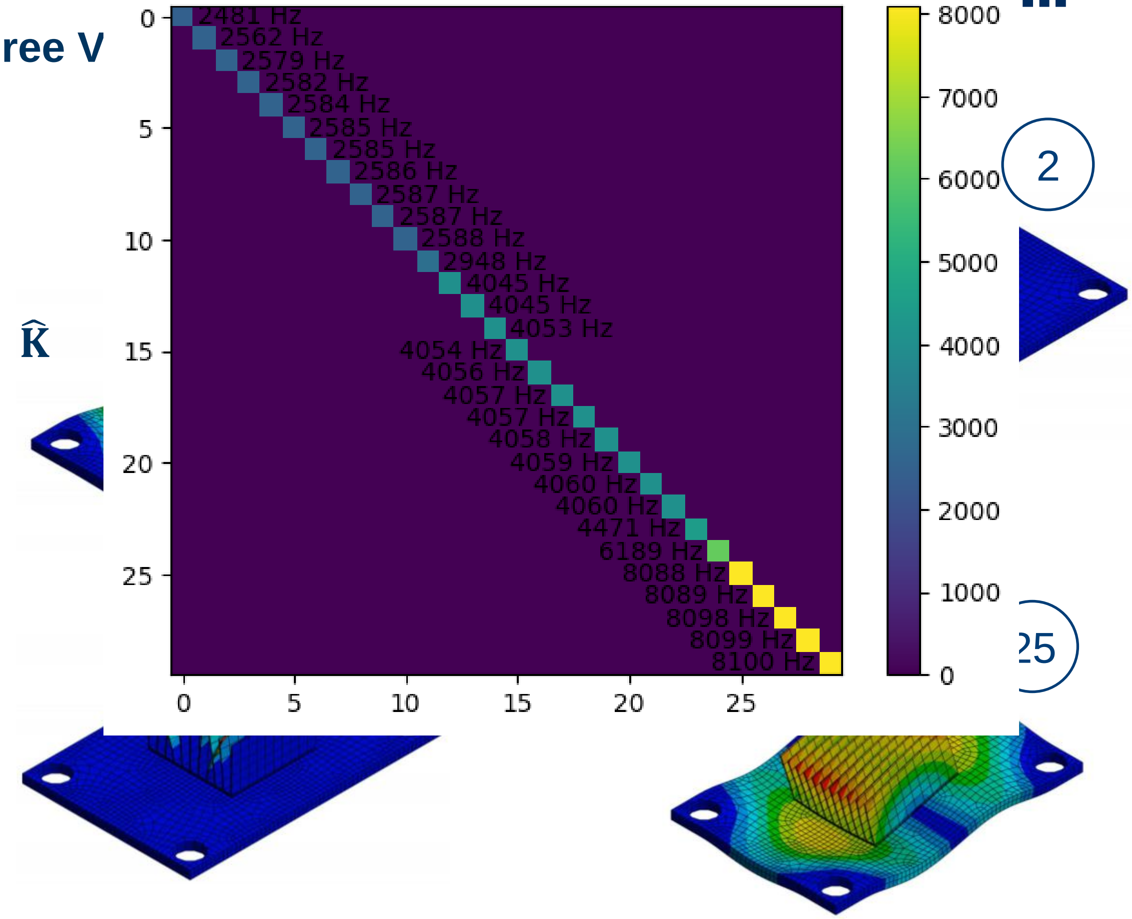
- Orthogonal basis
- System matrices ($\mathbf{K}$ and $\mathbf{M}$) are decoupled
  - Modal Superposition
  - With "Mass normalization"
  - $\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi} = \mathrm{diag}(1)$, scalar product
  - $\mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi} = \mathrm{diag}(\omega^2)$

$\hat{\mathbf{K}}$



Modal Analysis

“What the system wants to do” – Free V

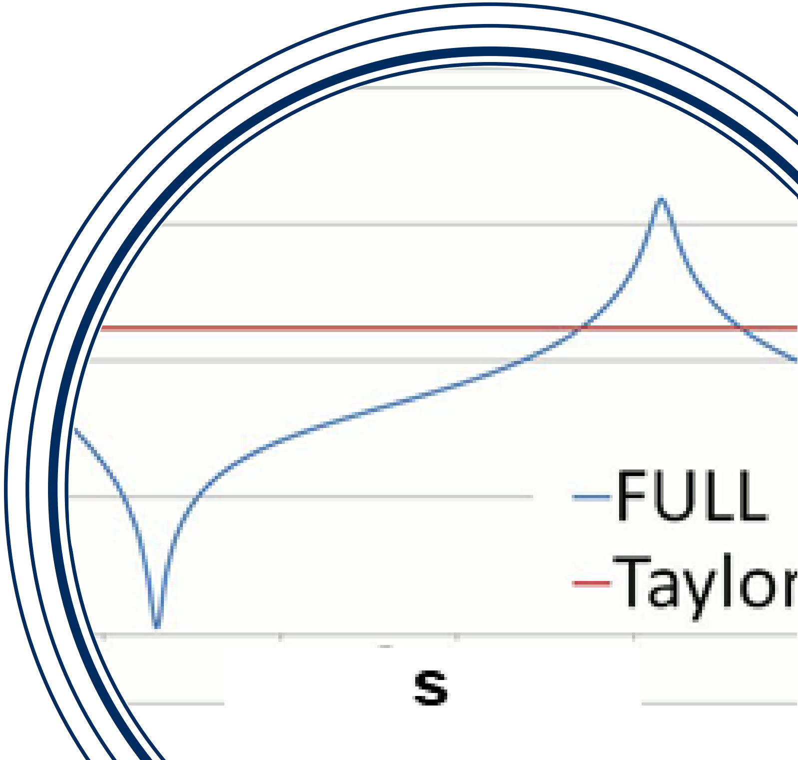


Drawbacks

- Small structures → many modes
- Excitation is not considered
  - Many inner modes
- Only for linear dynamics
  - Commonly used for structural

How to find a good Subspace

Krylov Subspace - Implicit Moment Matching



Krylov Subspace

General definition of Krylov Subspace

- $\mathcal{K} = \mathcal{K}(\mathbf{A}, \mathbf{q}) = \mathrm{span}\{\mathbf{q}, \mathbf{Aq}, \dots, \mathbf{A}^{m-1}\mathbf{q}\}$

Used in many algorithms

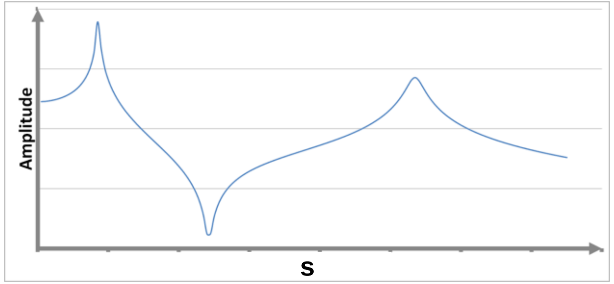
- Arnoldi (solution of eigenvalue problems)
- Lanczos (solution of eigenvalue problems)
- GMRES (approximate solution of systems of linear equations)
- QMR (approximate solution of systems of linear equations)
- …

Moment Matching

What is a Moment in this Context?

General transfer function

- in Laplace domain
- Fourier integral is a special case with $s = \mathrm{i}\omega$
- $(s^2\mathbf{M} - \mathbf{K})\mathbf{u} = \mathbf{F}_{\text{ext}}$
- $\underbrace{(s^2\mathbf{M} - \mathbf{K})}_{\mathbf{K}_{\text{eq}(s)}}{}^{-1}\mathbf{F}_{\text{ext}} = \mathbf{u}$



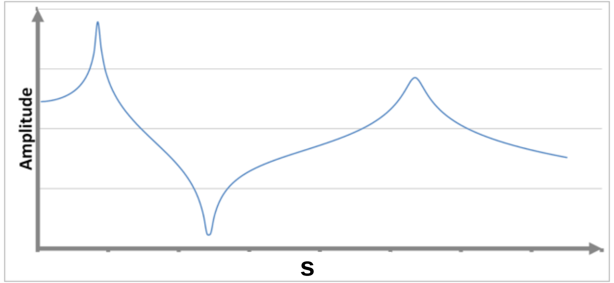
Moment Matching

What is a Moment in this Context?

$$\underbrace{(s^2\mathbf{M} - \mathbf{K})}_{\mathbf{K}_{\mathrm{eq}(s)}}^{-1} \mathbf{F}_{\mathrm{ext}} = \mathbf{u}$$

Taylor series

- around s_0^2
- $\mathbf{u} = \sum_i \; \mathbf{m}_i \, (s^2 - s_0^2)^i$
  - Hint: the i-th derivative in the Laplace domain is just $s^i \cdot f(s)$
  - $\mathbf{m}_i$ are called moments of the transfer function



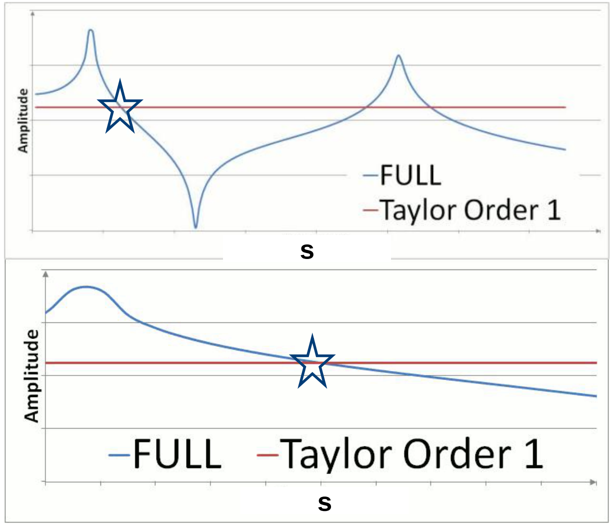
Moment Matching

Explicit Moment Matching

$$\mathbf{u} = \sum_i \mathbf{m}_i \left(s^2 - s_0^2\right)^i$$

Explicit calculation of moments is numerically unstable

- $\mathbf{m}_i = \widetilde{\mathbf{M}}^i \tilde{\mathbf{F}}$
- $\widetilde{\mathbf{M}}^i = \left[-\mathbf{K}_{\mathrm{eq}(s_0)}^{-1}\mathbf{M}\right]^i$
- $\tilde{\mathbf{F}} = \mathbf{K}_{\mathrm{eq}(s_0)}^{-1}\mathbf{F}_{\mathrm{ext}}$



Moment Matching

Implicit Moment Matching

Explicit Moments

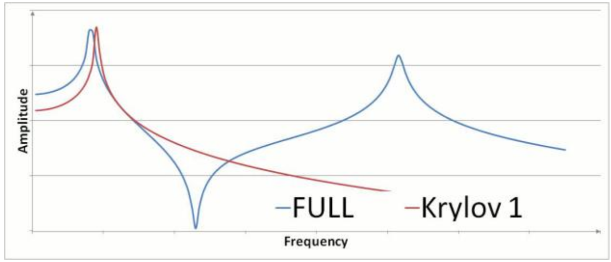
- $\mathbf{m}_i = \widetilde{\mathbf{M}}^i \widetilde{\mathbf{F}}$
- $\widetilde{\mathbf{M}}^i = \left[-\mathbf{K}_{\mathrm{eq}(s_0)}^{-1} \mathbf{M} \right]^i$
- $\widetilde{\mathbf{F}} = \mathbf{K}_{\mathrm{eq}(s_0)}^{-1} \mathbf{F}_{\mathrm{ext}}$

Krylov subspace definition

- $\mathcal{K} = \mathcal{K}(\mathbf{A}, \mathbf{q}) = \mathrm{span}\{\mathbf{q}, \mathbf{A}\mathbf{q}, \dots, \mathbf{A}^{m-1}\mathbf{q}\}$
- $\mathbf{A} = \widetilde{\mathbf{M}}$
- $\mathbf{q} = \widetilde{\mathbf{F}}$

The moments lie in the Krylov Subspace!

- The first moment $\mathbf{q}$ is just the solution at the expansion point

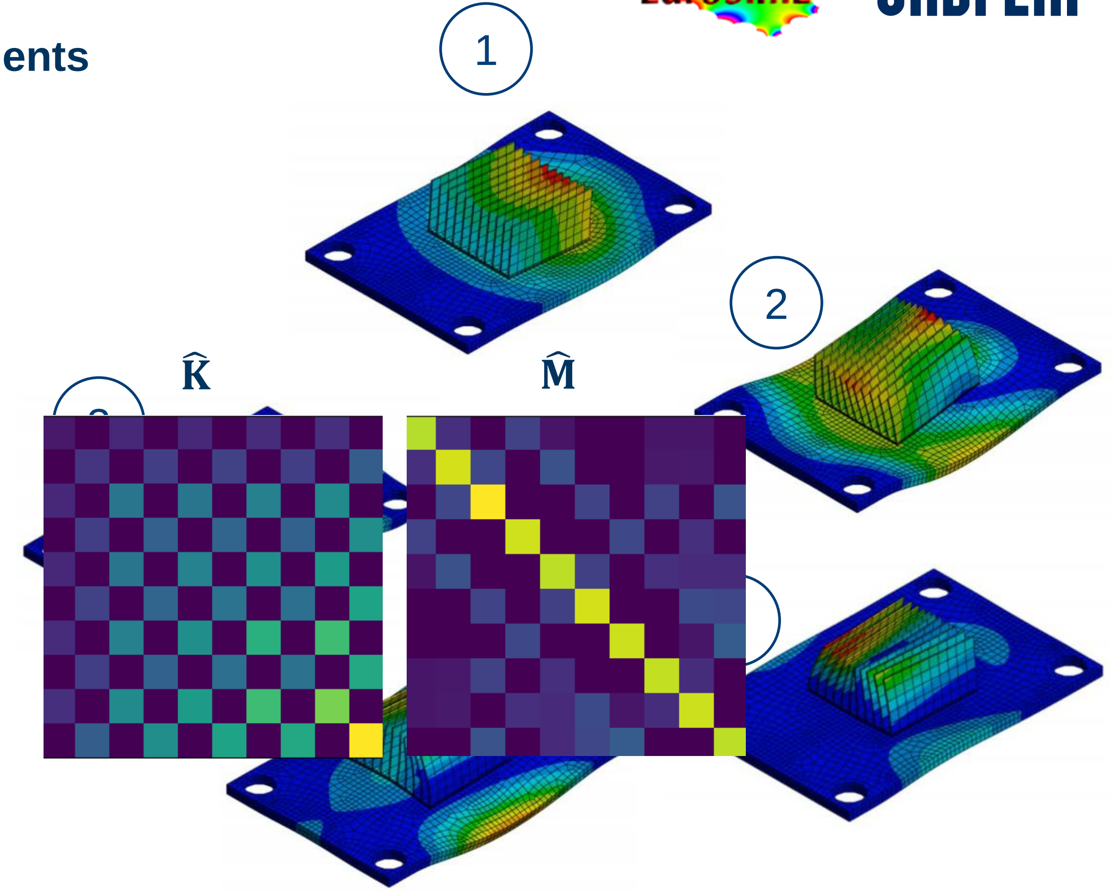


Moment Matching

Implicit Moment Matching - Comments

The load is included!

No decoupling of the reduced matrices



Moment Matching

Implicit Moment Matching - Comments

- Implicit moment matching is not limited to structural dynamics
- Expansion point should be set according to the frequency range of interest
- It is possible to use multiple expansion points
- There is no established rule how many vectors should be considered
- Can be expanded to “parametric model order reduction” – the Taylor expansion is then done also for the parameter

How to find a good Subspace

Based on Simulation Results

Methods to Determine Subspaces

Based on Simulation Results

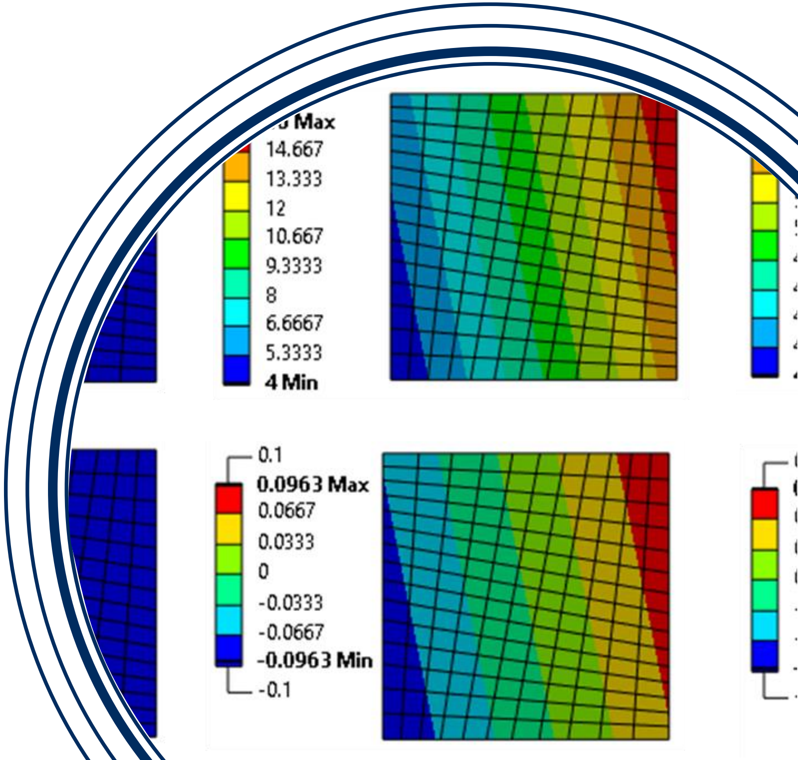
Snapshot matrix

- Do a calculation for m timesteps and assemble the results of interest $\mathbf{y}\ \mathbb{R}^N$ in a $\boldsymbol{S} = [\mathbf{y}_1, \ldots \mathbf{y}_n] \in \mathbb{R}^{N\times m}$

The snapshots define the vector space of the solution – how to find an orthonormal basis?

How to find a good Subspace

Gram Schmidt Orthogonalization



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Gram Schmidt Orthogonalization

EuroSimE CADFEM

Given

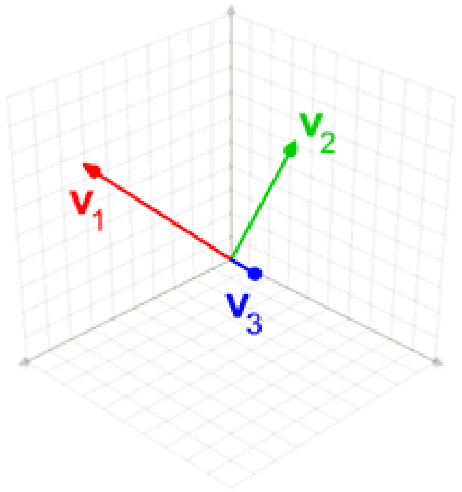
- m vectors $\mathbf{v}_i$ in $\mathbb{R}^N$

Target

- m orthonormal vectors $\mathbf{u}_i$ in $\mathbb{R}^N$

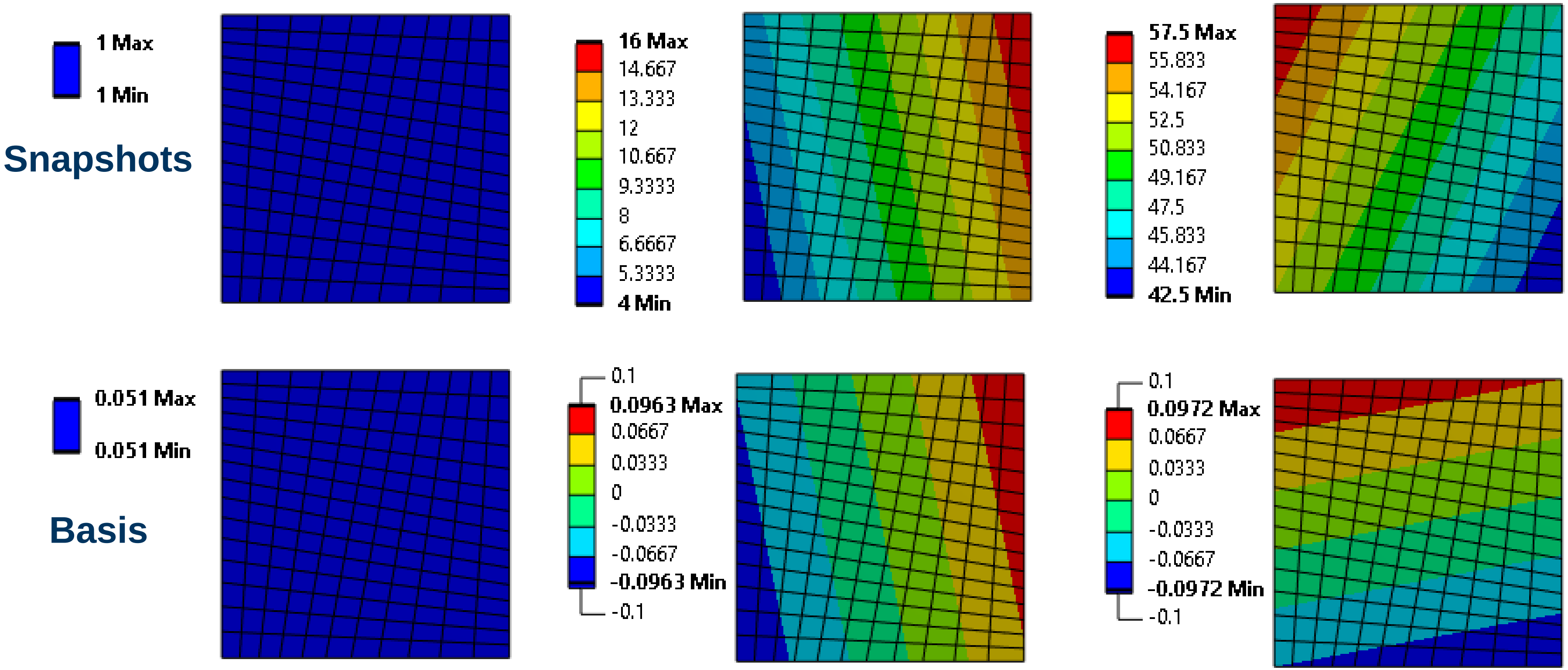
Algorithm

- $\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$ — Normalization of vector $\mathbf{v}_1$
- $\mathbf{u}_2' = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1$ — Orthogonalization of $\mathbf{v}_2$
- $\mathbf{u}_2 = \frac{\mathbf{u}_2'}{\|\mathbf{u}_2'\|}$ — Normalization of vector $\mathbf{u}_2'$
- $\mathbf{u}_3' = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2$ — Orthogonalization of $\mathbf{v}_3$
- $\mathbf{u}_3 = \frac{\mathbf{u}_3'}{\|\mathbf{u}_2'\|}$ — Normalization of vector $\mathbf{u}_3'$
- …
- $\mathbf{u}_m' = \mathbf{v}_m - \sum_{i=1}^{n-1} \frac{\langle \mathbf{v}_m, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$ — Orthogonalization of $\mathbf{v}_m$
- $\mathbf{u}_m = \frac{\mathbf{u}_m'}{\|\mathbf{u}_m'\|}$ — Normalization of vector $\mathbf{u}_m'$



https://upload.wikimedia.org/wikipedia/commons/e/ee/Gram-Schmidt_orthonormalization_process.gif

Gram Schmidt Orthogonalization



Gram Schmidt Orthogonalization

Applications

Find an orthonormal basis for given vectors

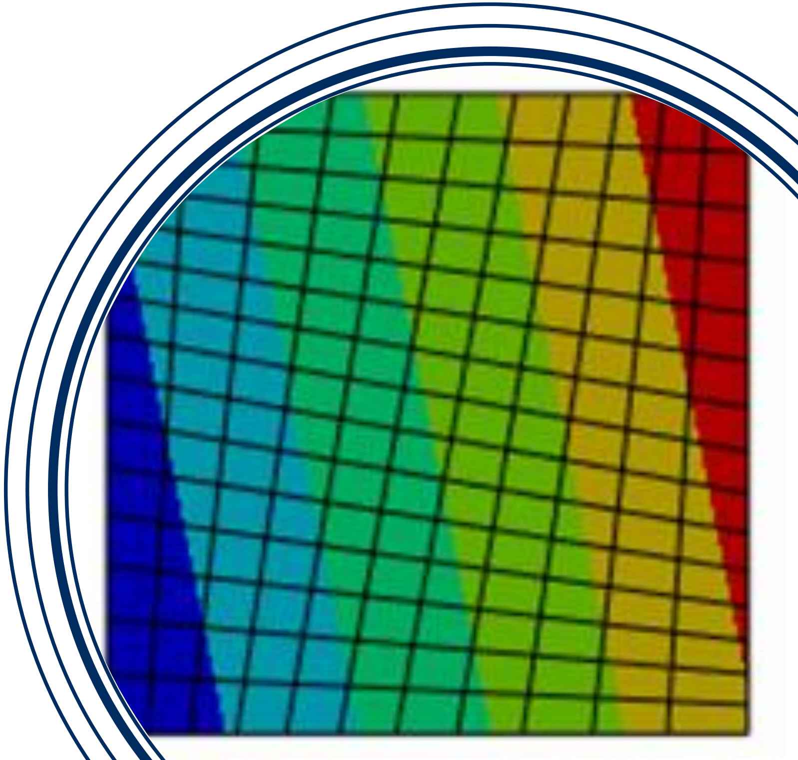
How to truncate the basis?

- Engineering Knowledge: Select appropriate snapshots to calculate the basis

Used in different numerical algorithms (e.g. for the calculation of Krylov subspaces)

How to find a good Subspace

Proper Orthogonal Decomposition



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Proper Orthogonal Decomposition

Target

Find the best linear subspace for the approximation of the data
$\mathbf{S} = [\mathbf{y}_1, \dots \mathbf{y}_n] \in \mathbb{R}^{N\times m}$

Principal Component Analysis

Compute the eigenvalue decomposition of $\mathbf{Y} = \mathbf{S}\mathbf{S}^{\mathrm{T}} \in \mathbb{R}^{N\times N}$
→ Sorted Eigenvalues $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_N\}$ and Eigenvectors $\boldsymbol{\Psi} = [\boldsymbol{\Psi}_1, \dots, \boldsymbol{\Psi}_N]$
→ Basisvectors $\boldsymbol{\Phi}_i = \boldsymbol{\Psi}_i \in \mathbb{R}^N$

Method of Snapshots

Compute the eigenvalue decomposition of $\mathbf{Y} = \mathbf{S}^{\mathrm{T}}\mathbf{S} \in \mathbb{R}^{m\times m}$
→ Sorted Eigenvalues $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_N\}$ and Eigenvectors $\boldsymbol{\Psi} = [\boldsymbol{\Psi}_1, \dots, \boldsymbol{\Psi}_N]$
→ Basisvectors $\boldsymbol{\Phi}_i = \mathbf{S}\boldsymbol{\Psi}_i \in \mathbb{R}^N$

Best Linear Approximation

Take n basisvectors corresponding to the largest eigenvalues

Proper Orthogonal Decomposition

Use Cases

Principal Component Analysis

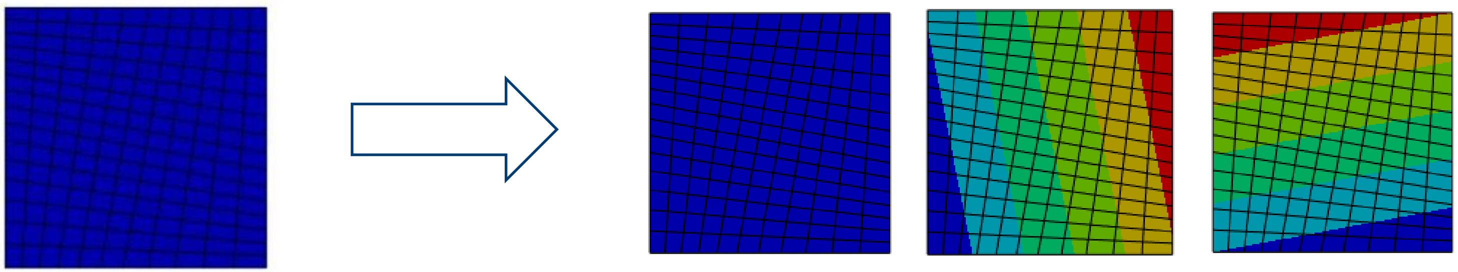
- Compute the eigenvalue decomposition of $\mathbf{Y} = \boldsymbol{S}\boldsymbol{S}^{\mathrm{T}} \quad \in \mathbb{R}^{N\times N}$
- Efficient if length N of snapshot vectors (e.g. locations) is low compared to count m of snapshots (e.g. timesteps)

→ Measurement data

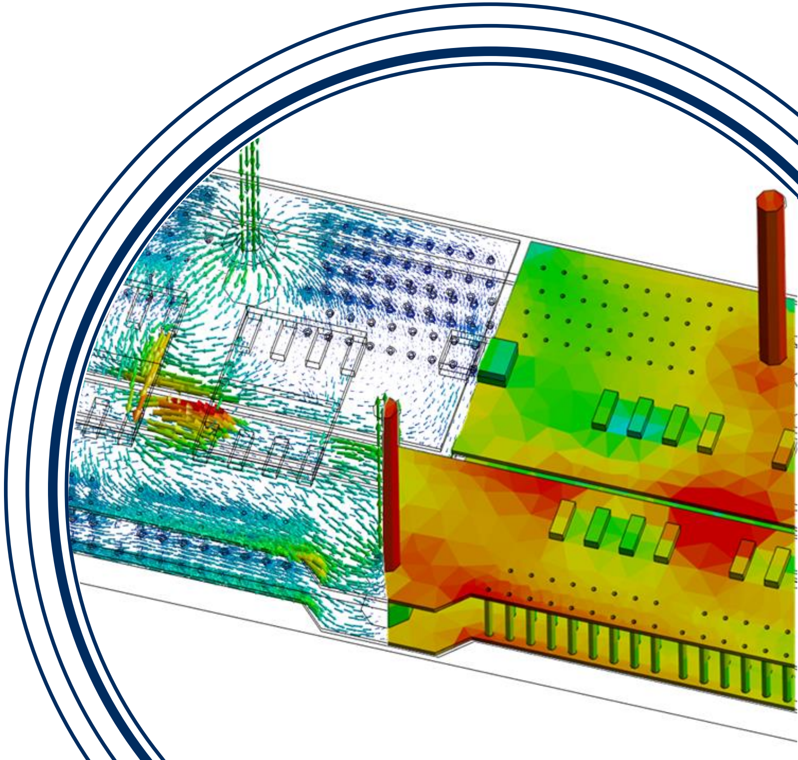
Method of Snapshots

- Compute the eigenvalue decomposition of $\mathbf{Y} = \boldsymbol{S}^{\mathrm{T}}\boldsymbol{S} \quad \in \mathbb{R}^{m\times m}$
- Efficient if count m of snapshots (e.g. timesteps) is low compared to length N of snapshot vectors (e.g. locations)

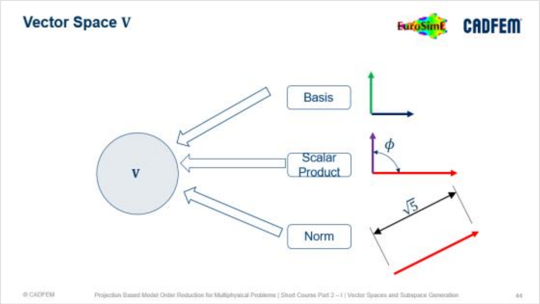
→ Simulation data

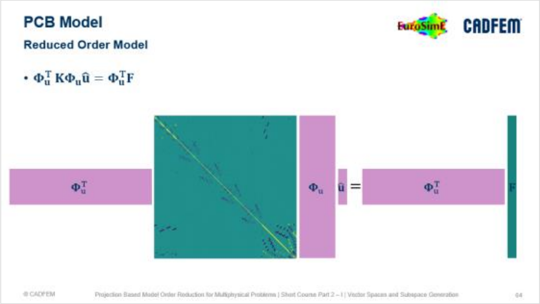


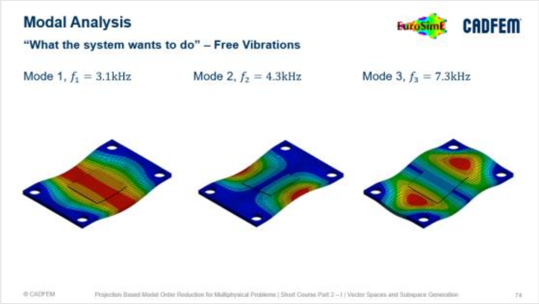
Summary

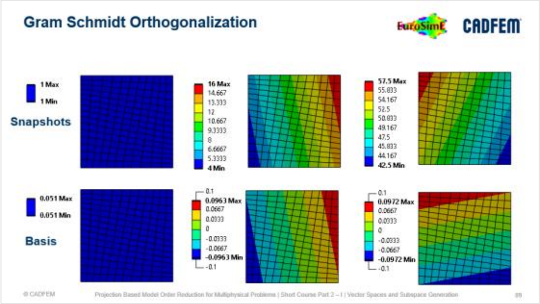


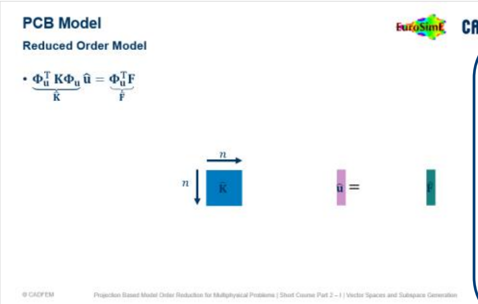
Summary

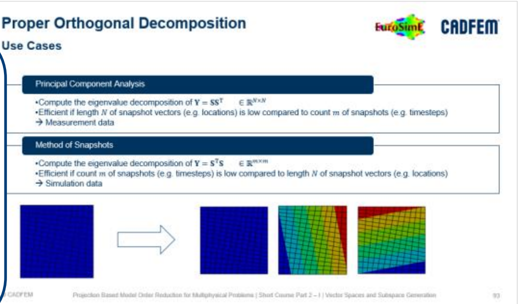












Nonlinear within ROM?

Session 18	Solder joint reliability simulation 1	10:45	Wednesday April 10 2024

11:05
20mn

Reduced-Order Model for Solder Balls – Potential of projection-based approaches for representing viscoplastic behavior

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